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KEZIA DA CRUZ BARROS

DECAIMENTO EXPONENCIAL DA ENERGIA EM UM MODELO DE MISTURA BINARIA NÃO AUTONOMO

BARCARENA

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Trabalho de conclusão apresentado ao colegiado do curso de Licenciatura em Matemática do Campus Universitário de Abaetetuba como um pré-requisito para a obtenção do grau de Licenciado em Matemática.

ORIENTADOR: PROF. DR. MANOEL JEREMIAS DOS SANTOS

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Exponential Energy Decay, Determining Functionals, and Numerical Simulations for a Nonlinear Non-Autonomous Binary Mixture Model

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Abstract

In this work, we investigate the asymptotic behavior of solutions to a nonlinear model describing binary mixtures of solids. We establish the exponential decay of the energy in both the autonomous case (in the absence of external forces) and the non-autonomous case (under the influence of time-dependent external forces). For the autonomous system, we further analyze the observability properties of the associated dynamical system by providing a sufficient upper bound on the completeness defect that ensures a given set of bounded linear functionals constitutes a determining set. Numerical experiments are performed using the finite element method and Newmark's algorithm to illustrate the conservative and dissipative behaviors of the proposed nonlinear coupled model. The results confirm the accuracy of the scheme and the exponential decay of the energy in dissipative cases.

Keywords: Non-Autonomous binary mixture, well-posedness, decay energy, determining functionals, finite element method, Newmark algorithm.

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1 Introduction

The idea of modeling a material as a homogeneous continuum has proven to be extremely effective in various areas of mechanics, such as hydrodynamics, aerodynamics, rigid body dynamics, and plasticity, maintaining its relevance to the present day [18, 32, 35].

However, in many contexts, the interest lies precisely in the behavior of the material at the scale of its inhomogeneities and in the states of its individual constituents. To address this challenge, researchers in mechanics developed two innovative and fruitful concepts: the continuum theory of mixtures and the theory of materials with microstructure. The modern formulation of continuum theory for mixtures began with the seminal works presented in [8, 13, 19, 20, 23, 24, 28, 35], and comprehensive expositions of this theory can be found, for example, in [4, 6, 7, 31].

In the continuum theory of mixtures, the fundamental concept is to consider that the constituents of a mixture can be represented by superposed continua, so that each point of the mixture is simultaneously occupied by a material point of each constituent [6].

Let us consider a mixture composed of two elastic materials, $u = s_1$ and $w = s_2$, with initial (reference) configuration in the interval $[0, 1]$. We denote by ρ_i the density, by T_i the partial stress,

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by F_i the external force, and by P_i the internal force associated with the material s_i. Assuming that $u = u(x,t)$ and $w = w(x,t)$ represent the transverse displacements of the particle x in materials u and w, respectively, at time t, we obtain the following equations (see [22, 7, 26, 35])

$$\rho_1 u_{tt} = T_{1x} - P_1 + F_1, \tag{1.1}$$
$$\rho_2 w_{tt} = T_{2x} + P_2 + F_2. \tag{1.2}$$

We assume that the partial stresses are given by

$$T_1 = a_1 u_x + a_2 w_x, \tag{1.3}$$
$$T_2 = a_2 u_x + a_3 w_x, \tag{1.4}$$

where

$$A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$$

is a symmetric (real) positive definite matrix. We also assume that the internal and external forces take the form

$$P_1 = f_1(u,w) + g_1(u_t), \quad P_2 = -f_2(u,w) - g_2(w_t), \quad F_1 = h_1, \quad F_2 = h_2. \tag{1.5}$$

Substituting (1.3)–(1.4) and (1.5) into (1.1)–(1.2), we obtain the system

$$\rho_1 u_{tt} - a_1 u_{xx} - a_2 w_{xx} + g_1(u_t) + f_1(u,w) = h_1(x,t), \quad \text{in } (0,1)\times(0,\infty), \tag{1.6}$$
$$\rho_2 w_{tt} - a_2 u_{xx} - a_3 w_{xx} + g_2(w_t) + f_2(u,w) = h_2(x,t), \quad \text{in } (0,1)\times(0,\infty), \tag{1.7}$$

with boundary conditions

$$u(0,t) = u(1,t) = w(0,t) = w(1,t) = 0, \quad \forall t > 0, \tag{1.8}$$

and initial conditions

$$\begin{aligned} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \forall x \in [0,1], \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad \forall x \in [0,1], \end{aligned} \tag{1.9}$$

where u_0, u_1, w_0, and w_1 are known functions to be specified later. We assume that $a_1 > 0$, $a_2 \in \mathbb{R}$, $a_3 > 0$, and that

$$a_1 a_3 - a_2^2 > 0. \tag{1.10}$$

The asymptotic behavior of solutions to linear and nonlinear models based on the continuum theory of mixtures — including gases, fluids, and solids — has been extensively investigated in recent decades [1, 2, 3, 12, 15, 16, 17, 26, 29, 33]. However, the number of studies involving nonlinear models is still significantly lower than those dedicated to linear models.

Freitas et al. [15] analyzed the case where $P_1 = F_1 = F_2 = 0$ and $P_2 = -\mu u_t - f(u)$. More precisely, they studied the system

$$\begin{aligned} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \quad \text{in } (0,L)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \mu u_t + f(u) = 0, \quad \text{in } (0,L)\times(0,\infty), \end{aligned} \tag{1.11}$$

with boundary conditions

$$z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = 0, \quad \forall t > 0. \tag{1.12}$$

It was shown that the dynamical system generated by the solutions admits a compact global attractor with finite fractal dimension.

Santos et al. [33] studied the asymptotic behavior of (1.6)–(1.9) with $h_i \equiv 0$ for $i = 1,2$. It was shown that if the initial energy is negative, then every weak solution blows up in finite time. Moreover, using the Nehari manifold approach, the authors proved the existence of a unique global weak solution when the initial data belong to the "good part" of the potential well. For these solutions, the total energy of the system decays either exponentially or algebraically, depending on the behavior of the dissipation near the origin.

Miranda et al. [27] considered a nonlinear model with a fractional delay term based on the continuum theory of mixtures, analyzing the qualitative properties of the system

$$\begin{aligned} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \quad \text{in } (0,l)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + (-\partial_{xx})^\theta u_t + \varepsilon(-\partial_{xx})^\theta u_t(x,t-\tau) + f(u) = h, \quad \text{in } (0,l)\times(0,\infty), \end{aligned} \tag{1.13}$$

with boundary conditions (1.12). The existence of a family of global attractors indexed by $\varepsilon > 0$ was proved, and it was shown that, in the limit $\varepsilon \to 0$, these attractors converge to the global attractor of the limiting system

$$\begin{aligned} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \quad \text{in } (0,l)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + (-\partial_{xx})^\theta u_t + f(u) = h, \quad \text{in } (0,l)\times(0,\infty). \end{aligned} \tag{1.14}$$

Our investigations indicate that works considering exponential energy decay in nonlinear models based on continuum mixture theory are scarce. Moreover, to date, we have not found studies exploring the use of determining functionals in this context.

The hypothesis that the asymptotic dynamics of dissipative systems governed by partial differential equations (PDEs) can be described by a finite number of parameters has motivated the development of several dimensionality reduction approaches. Among them, the concept of determining functionals stands out, introduced in the works of FOIAS and PRODI [14], and LADYZHENSKAYA [25]. A finite set of linear and continuous functionals is said to be determining when the convergence of their values over time implies the convergence of the corresponding trajectories in phase space, thus allowing the asymptotic behavior of the system to be described from a finite number of scalar observables.

This approach provides a robust alternative to the construction of inertial manifolds, whose existence often depends on restrictive assumptions. Determining functionals have been successfully applied to models such as the two-dimensional Navier–Stokes equations, in which several classes of functionals — such as Fourier modes, nodal values, and local averages — have been shown to capture the asymptotic dynamics of the system.

In this work, we apply the theory of determining functionals to a nonlinear model of binary mixture. Our goal is to identify functional subsets that ensure the asymptotic uniqueness of the system's trajectories, as well as to provide explicit estimates for the minimal number of observations required. This analysis contributes to the theoretical understanding of the dynamic behavior of flexible structures under nonlinear dissipative effects.

The paper is organized as follows: in Section 2, we discuss the functional spaces relevant to the model, providing definitions and fundamental properties. Section 3 presents the assumptions concerning the source and damping terms. In Section 4, we introduce the phase space and demonstrate the well-posedness of the problem, including the existence and uniqueness of solutions, as well as the definition of the dynamical system $(\mathscr{H}, S(t))$. In Section 5, we study the exponential decay of the energy of solutions in both autonomous and non-autonomous cases. Section 6 is devoted to the investigation of determining functionals for the system $(\mathscr{H}, S(t))$, establishing the completeness defect and showing that a finite set of bounded linear functionals can serve as determining functionals. Finally, in Section 7, we formulate the corresponding variational problem and derive its finite element semi-discretization in space combined with a finite difference discretization in time. The fully discrete system is obtained through the well-known Newmark integration scheme, and an iterative algorithm is implemented to handle the nonlinear terms. Several numerical experiments are then performed to validate the theoretical results, illustrating the conservative and dissipative behaviors of the system, as well as the exponential decay rate of the total energy in the presence of damping.

2 Functional Spaces

For $1 \leqslant p < \infty$, we denote by $L^p(0,1)$ the Lebesgue space of measurable functions $u : [0,1] \to \mathbb{R}$ such that the p-th power of their modulus is integrable on $[0,1]$. The norm in this space is denoted by $\|\cdot\|_p$, that is,

$$\|u\|_p = \left(\int_0^1 |u|^p \right)^{1/p}. \tag{2.1}$$

For $p = \infty$, we have

$$\|u\|_\infty = \inf\{C > 0;\ |u(x)| \leqslant C, \text{ a.e. in } [0,1]\}. \tag{2.2}$$

In particular, the inner product and norm in $L^2(0,1)$ are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. In this case,

$$\langle u, v \rangle := \int_0^1 uv dx \quad \text{and} \quad \|u\|^2 = \langle u, u \rangle, \quad \forall u, v \in L^2(0,1). \tag{2.3}$$

In this paper, we will also use the following Sobolev spaces based on $L^2(0,1)$:

$$H^1(0,1) \quad = \quad \{u \in L^2(0,1);\ u_x \in L^2(0,1)\}, \tag{2.4}$$

$$H^2(0,1) \quad = \quad \{u \in L^2(0,1);\ u_x, u_{xx} \in L^2(0,1)\}, \tag{2.5}$$

where u_x and u_{xx} denote the weak first and second derivatives of u, respectively. Moreover, we also consider the space

$$H_0^1(0,1) \quad = \quad \{u \in H^1(0,1);\ u(0) = u(1) = 0\}. \tag{2.6}$$

It follows from Poincaré's inequality, that is,

$$\|u\| \leqslant \|u_x\|, \quad \forall u \in H_0^1(0,1), \tag{2.7}$$

that we can use the following norm in $H_0^1(0,1)$:

$$\|u\|_{H_0^1(0,1)} = \|u_x\|, \quad \forall u \in H_0^1(0,1). \tag{2.8}$$

The dual space of $H_0^1(0,1)$ is denoted by $H^{-1}(0,1)$.

3 Assumptions

Concerning the terms f_i, g_i, and h_i, for $i = 1, 2$, we impose the following conditions:

($\mathbf{A}_1$) $g_i \in C^1(\mathbb{R})$, $g_i(0) = 0$, and there exist constants m and M such that

$$m \leqslant g_i'(s) \leqslant M, \quad \forall s \in \mathbb{R}. \tag{3.1}$$

From the inequality above and the Mean Value Theorem, it follows that

$$m|s - r|^2 \leqslant (g_i(s) - g_i(r))(s - r) \leqslant M|s - r|^2, \quad \forall s, r \in \mathbb{R}. \tag{3.2}$$

($\mathbf{A}_2$) There exists $F : \mathbb{R}^2 \to \mathbb{R}$, with $F \in C^2(\mathbb{R}^2)$ such that

$$\nabla F = (f_1, f_2) \tag{3.3}$$

$$\nabla F(u, w) \cdot (u, w) \geqslant F(u, w) \geqslant 0, \quad \forall (u, w) \in \mathbb{R}^2, \tag{3.4}$$

and there exist $p \geqslant 1$ and $C_f > 0$ such that

$$|\nabla f_i(u, w)| \leqslant C_f(1 + |u|^p + |w|^p); \tag{3.5}$$

It follows from the previous inequality that there exists a positive constant C_F such that

$$F(u, w) \leqslant C_F(1 + |u|^{p+1} + |w|^{p+1}), \quad \forall (u, w) \in \mathbb{R}^2. \tag{3.6}$$

($\mathbf{A}_3$) $h_i \in L^2_{loc}([0,\infty); L^2(0,1))$

($\mathbf{A}_4$) For exponential decay, we further assume that there exist positive constants σ and C_h such that

$$\int_0^\infty e^{\sigma s}(\|h_1(s)\|^2 + \|h_2(s)\|^2) ds < C_h. \tag{3.7}$$

4 Well-Posedness

We consider the following Hilbert space

$$\mathscr{H} = \mathscr{V}_1 \times \mathscr{V}_0, \tag{4.1}$$

where

$$\mathscr{V}_1 = \left(H_0^1(0,1)\right)^2 \quad \text{and} \quad \mathscr{V}_0 = \left(L^2(0,1)\right)^2. \tag{4.2}$$

The inner product in $\mathscr{H}$ is given by

$$\begin{aligned}\langle (u,w,v,z),(\phi,\psi,\varphi,\chi)\rangle_{\mathscr{H}} := \rho_1\langle v,\varphi\rangle + \rho_2\langle z,\chi\rangle + \left\langle \sqrt{a_1}u_x + \frac{a_2}{\sqrt{a_1}}w_x, \sqrt{a_1}\phi_x + \frac{a_2}{\sqrt{a_1}}\psi_x \right\rangle \\ + \left(a_3 - \frac{a^2}{a_1}\right)\langle w_x,\psi_x\rangle\end{aligned} \tag{4.3}$$

and the induced norm

$$\|(u,w,v,z)\|_{\mathscr{H}}^2 = \rho_1\|v\|^2 + \rho_2\|z\|^2 + \left\|\sqrt{a_1}u_x + \frac{a_2}{\sqrt{a_1}}w_x\right\|^2 + \left(a_3 - \frac{a_2^2}{a_1}\right)\|w_x\|^2. \tag{4.4}$$

The usual norm in $\mathscr{H}$ is denoted by $\|\cdot\|_*$, that is,

$$\|(u,w,v,z)\|_*^2 = \|v\|^2 + \|z\|^2 + \|u_x\|^2 + \|w_x\|^2. \tag{4.5}$$

In $\mathscr{V}_1$ we consider the norm

$$\|(u,w)\|_{\mathscr{V}_1}^2 := \left\|\sqrt{a_1}u_x + \frac{a_2}{\sqrt{a_1}}w_x\right\|^2 + \left(a_3 - \frac{a_2^2}{a_1}\right)\|w_x\|^2 \tag{4.6}$$

and the usual norm

$$\|(u,w)\|_\star^2 := \|u_x\|^2 + \|w_x\|^2. \tag{4.7}$$

It is easy to prove that $\|\cdot\|_{\mathscr{V}_1}$ and $\|\cdot\|_\star$ are equivalent, that is, there exist positive constants σ_0 and σ_1 such that

$$\sigma_0\|(u,w)\|_\star^2 \leqslant \|(u,w)\|_{\mathscr{V}_1}^2 \leqslant \sigma_1\|(u,w)\|_\star^2 \quad \forall (u,w) \in \mathscr{V}_1, \tag{4.8}$$

where

$$\sigma_0 = \min\left\{\frac{a_1}{2}, \left(\frac{2a_2^2}{a_1^2}+1\right)^{-1}\left(a_3 - \frac{a_2^2}{a_1}\right)\right\} \quad \text{and} \quad \sigma_1 = \max\left\{2a_1, a_3 + \frac{a_2^2}{a_1}\right\}.$$

The previous equivalence implies the equivalence of $\|\cdot\|_{\mathscr{H}}$ and $\|\cdot\|_*$ and hence, there exist positive constants γ_0 and γ_1 such that

$$\gamma_0\|(u,w,v,z)\|_*^2 \leqslant \|(u,w,v,z)\|_{\mathscr{H}}^2 \leqslant \gamma_1\|(u,w,v,z)\|_*^2, \quad \forall (u,w,v,z) \in \mathscr{H}, \tag{4.9}$$

where

$$\gamma_0 = \min\{\rho_1,\rho_2,\sigma_0\} \quad \text{and} \quad \gamma_1 = \max\{\rho_1,\rho_2,\sigma_1\}.$$

Once the phase space $\mathscr{H}$ is established, considering

$$U(t) = (u(t), w(t), u_t(t), w_t(t)) \quad \text{and} \quad U_0 = (u_0, w_0, u_1, w_1) \tag{4.10}$$

we can rewrite (1.6)-(1.9) as the following initial value problem:

$$\begin{cases} U_t(t) + \mathcal{A}U(t) + \mathcal{B}(U(t)) = \mathcal{F}(t), \quad t > 0 \\ U(0) = U_0, \end{cases} \tag{4.11}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathscr{H} \to \mathscr{H}$ is given by

$$\mathcal{A}\begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} = \begin{pmatrix} -v \\ -z \\ \rho_1^{-1}\big(-a_1u_{xx} - a_2w_{xx}\big) \\ \rho_2^{-1}\big(-a_2u_{xx} - a_3w_{xx}\big) \end{pmatrix} \tag{4.12}$$

with

$$D(\mathcal{A}) := \big(H_0^1(0,1) \cap H^2(0,1)\big)^2 \times \big(H_0^1(0,1)\big)^2, \tag{4.13}$$

$\mathcal{B} : \mathscr{H} \to \mathscr{H}$ is given by

$$\mathcal{B}\begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \rho_1^{-1}\big(g_1(v) + f_1(u,w)\big) \\ \rho_2^{-1}\big(g_2(z) + f_2(u,w)\big) \end{pmatrix} \tag{4.14}$$

and $\mathcal{F} : \mathbb{R}^+ \to \mathscr{H}$ is given by

$$\mathcal{F}(t) = \begin{pmatrix} 0 \\ 0 \\ \rho_1^{-1}h_1(x,t) \\ \rho_2^{-1}h_2(x,t) \end{pmatrix}. \tag{4.15}$$

Lemma 4.1. The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathscr{H}$, denoted by $\{T(s)\}_{t\geqslant 0}$.

Proof. First, note that $\mathcal{A}$ is dissipative, since

$$\langle \mathcal{A}W, W\rangle_{\mathscr{H}} = 0, \quad \forall\, W = (u,w,v,z) \in D(\mathcal{A}). \tag{4.16}$$

Moreover, we have $R(I-\mathcal{A}) = \mathscr{H}$, where $R(I-\mathcal{A})$ denotes the range of the operator $I-\mathcal{A}$. Indeed, given $W^* = (u^*, w^*, v^*, z^*) \in \mathscr{H}$, we will show that there exists $W = (u,w,v,z) \in D(\mathcal{A})$ such that

$$W - \mathcal{A}W = W^*. \tag{4.17}$$

From the previous equality, we obtain

$$u - v = u^*, \tag{4.18}$$
$$w - z = w^*, \tag{4.19}$$
$$\rho_1 v - a_1u_{xx} - a_2w_{xx} = \rho_1 v^*, \tag{4.20}$$
$$\rho_2 z - a_2u_{xx} - a_3w_{xx} = \rho_2 z^*. \tag{4.21}$$

Substituting (4.18) into (4.20) and (4.19) into (4.21), we obtain

$$\rho_1 u - a_1u_{xx} - a_2w_{xx} = \rho_1(v^* + u^*), \tag{4.22}$$
$$\rho_2 w - a_2u_{xx} - a_3w_{xx} = \rho_2(z^* + w^*). \tag{4.23}$$

A weak solution of (4.22)-(4.23) is a pair $(u,w) \in \mathscr{V}_1$ such that

$$\Xi\big((u,w),(\varphi,\psi)\big) = \Phi_*(\varphi,\psi), \quad \forall(\varphi,\psi) \in \mathscr{V}_1, \tag{4.24}$$

where Ξ is the sesquilinear form on $\mathscr{V}_1$ given by

$$\begin{aligned} \Xi\big((u,w),(\varphi,\psi)\big) &:= \rho_1\int_0^1 u\varphi dx + a_1\int_0^1 u_x\varphi_x dx + \rho_2\int_0^1 w\psi dx + a_3\int_0^1 w_x\psi_x dx \\ &+ a_2\int_0^1 (w_x\varphi_x + u_x\psi_x)dx \end{aligned} \tag{4.25}$$

and Φ_* is the linear form on $\mathscr{V}_1$ given by

$$\Phi_*(\varphi,\psi) = \rho_1 \int_0^1 (v^* + u^*)\varphi dx + \rho_2 \int_0^1 (z^* + w^*)\psi dx. \tag{4.26}$$

Using the norm $\|\cdot\|_{\mathscr{V}_1}$, it is easy to show that Ξ is continuous and coercive, and Φ_* is bounded. Therefore, it follows from the Lax-Milgram Theorem that (4.22)-(4.23) has a unique weak solution (u,w). Multiplying (4.22) by a_3 and (4.23) by $-a_2$ yields $u_{xx} \in L^2(0,1)$ and hence $u \in H^2(0,1)$, and using (4.22) or (4.23) we obtain $w \in H^2(0,1)$. Using the test functions $C_c^\infty(0,1)$, we obtain that u and w satisfy (4.22)-(4.23) almost everywhere in $[0,1]$, and defining

$$v = u - u^* \quad \text{and} \quad z = w - w^*, \tag{4.27}$$

we have $v, z \in H_0^1(0,1)$. Therefore, $W = (u,w,v,z) \in D(\mathcal{A})$ and satisfies (4.17). The lemma follows from the Lumer-Phillips Theorem, and this concludes the proof. □

Corollary 4.1. The operator $\mathcal{A}$ is m-accretive in $\mathscr{H}$.

Lemma 4.2. Under assumptions $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$, the operator $\mathcal{B}$ is locally Lipschitz continuous.

Proof. Given $K > 0$, for any $W_i = (u^i, w^i, v^i, z^i) \in \mathscr{H}$, $i = 1,2$ such that

$$\|W_i\|_{\mathscr{H}} \leqslant K, \quad i = 1,2, \tag{4.28}$$

we have

$$\begin{aligned} \|\mathcal{B}(W_1) - \mathcal{B}(W_2)\|_{\mathscr{H}}^2 =& \rho_1^{-1} \int_0^1 |g_1(v^1) - g_1(v^2) + f_1(u^1,w^1) - f_1(u^2,w^2)|^2 dx \\ &+ \rho_2^{-1} \int_0^1 |g_2(z^1) - g_2(z^2) + f_2(u^1,w^1) - f_2(u^2,w^2)|^2 dx. \end{aligned} \tag{4.29}$$

Note that

$$\begin{aligned} &\rho_1^{-1} \int_0^1 |g_1(v^1) - g_1(v^2) + f_1(u^1,w^1) - f_1(u^2,w^2)|^2 dx \\ &\leqslant 2\rho_1^{-1} \int_0^1 |g_1(v^1) - g_1(v^2)|^2 dx + 2\rho_1^{-1} \int_0^1 |f_1(u^1,w^1) - f_1(u^2,w^2)|^2 dx. \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} &\rho_2^{-1} \int_0^1 |g_2(z^1) - g_2(z^2) + f_2(u^1,w^1) - f_2(u^2,w^2)|^2 dx \\ &\leqslant 2\rho_2^{-1} \int_0^1 |g_2(z^1) - g_2(z^2)|^2 dx + 2\rho_2^{-1} \int_0^1 |f_2(u^1,w^1) - f_2(u^2,w^2)|^2 dx. \end{aligned} \tag{4.31}$$

It follows from (3.2) that

$$2\rho_1^{-1} \int_0^1 |g_1(v^1) - g_1(v^2)|^2 dx \leqslant 2\rho_1^{-2} M \rho_1 \|v^1 - v^2\|^2 \leqslant 2\rho_1^{-2} M \|W_1 - W_2\|_{\mathscr{H}}^2. \tag{4.32}$$

Similarly,

$$2\rho_2^{-1} \int_0^1 |g_2(z^1) - g_2(z^2)|^2 dx \leqslant 2\rho_2^{-2} M \|W_1 - W_2\|_{\mathscr{H}}^2. \tag{4.33}$$

On the other hand, since f_1 is of class C^1, it follows from the mean value inequality, from (3.5), and from Hölder's inequality that

$$\begin{aligned} &2\rho_1^{-1} \int_0^1 |f_1(u^1,w^1) - f_1(u^2,w^2)|^2 dx \\ &\leqslant C \int_0^1 \left[\sum_{j=1}^2 \left(1 + |u^j|^{2(p-1)} + |w^j|^{2(p-1)} \right) \right] |(u^1 - u^2, w^1 - w^2)|^2 dx \\ &\leqslant C \left[\sum_{j=1}^2 \left(1 + \|u^j\|_{2p}^{2(p-1)} + \|w^j\|_{2p}^{2(p-1)} \right) \right] \left(\|u^1 - u^2\|_{2p}^2 + \|w^1 - w^2\|_{2p}^2 \right). \end{aligned} \tag{4.34}$$

Using the embedding $H_0^1(0,1) \hookrightarrow L^{2p}(0,1)$ and the equivalence (4.8), we get

$$\begin{aligned}
&\left[\sum_{j=1}^{2}\left(1+\|u^j\|_{2p}^{2(p-1)}+\|w^j\|_{2p}^{2(p-1)}\right)\right]\left(\|u^1-u^2\|_{2p}^2+\|w^1-w^2\|_{2p}^2\right)\\
&\leqslant \left[\sum_{j=1}^{2}\left(1+\|u_x^j\|^{2(p-1)}+\|w_x^j\|^{2(p-1)}\right)\right]\left(\|u_x^1-u_x^2\|^2+\|w_x^1-w_x^2\|^2\right)\\
&\leqslant C\left(1+\|W_1\|_{\mathscr{H}}^{2(p-1)}+\|W_2\|_{\mathscr{H}}^{2(p-1)}\right)\|W_1-W_2\|_{\mathscr{H}}^2\\
&\leqslant C\left(1+K^{2(p-1)}+K^{2(p-1)}\right)\|W_1-W_2\|_{\mathscr{H}}^2\\
&\leqslant C(K)\|W_1-W_2\|_{\mathscr{H}}^2,
\end{aligned} \tag{4.35}$$

where $C(K)$ is a (generic) positive constant depending on K. From (4.34) and (4.35), we obtain

$$2\rho_1^{-1}\int_0^1 |f_1(u^1,w^1)-f_1(u^2,w^2)|^2 dx \leqslant C(K)\|W_1-W_2\|_{\mathscr{H}}^2. \tag{4.36}$$

Similarly,

$$2\rho_2^{-1}\int_0^1 |f_2(u^1,w^1)-f_2(u^2,w^2)|^2 dx \leqslant C(K)\|W_1-W_2\|_{\mathscr{H}}^2. \tag{4.37}$$

From (4.29), (4.32)-(4.33), and (4.36)-(4.37), we obtain

$$\left\|\mathcal{B}(W_1)-\mathcal{B}(W_2)\right\|_{\mathscr{H}}^2 \leqslant \big(C+C(K)\big)\|W_1-W_2\|_{\mathscr{H}}^2, \tag{4.38}$$

where C is a (generic) constant that does not depend on K, and this concludes the proof of the lemma. □

In order to establish the well-posedness of the initial value problem (4.11), we will use the theory of m-accretive operators in Hilbert spaces (see [5, 34], [11, Chapter II]).

Definition 4.1 (**Strong Solution**). A strong solution of (4.11) on $[0,T]$ is a continuous function $U:[0,T]\to\mathscr{H}$, which is absolutely continuous on every interval $[a,b]$, $0<a<b<T$, hence U is differentiable a.e. with $\frac{dU}{dt}\in L^1(0,T;\mathscr{H})$, $U(t)\in D(\mathcal{A})$ for all $t\in[0,T]$, and (4.11) is satisfied. ■

Definition 4.2 (**Generalized Solution**). A generalized solution of (4.11) on $[0,T]$ is a function $U\in C([0,T];\mathscr{H})$ such that $U(0)=U_0$ and there exists a sequence of strong solutions (U_n) defined on $[0,T]$ for the problem

$$\frac{dU_n}{dt}+\mathcal{A}U_n+\mathcal{B}(U_n)=\mathcal{F}_n, \quad n\geqslant 1 \tag{4.39}$$

with $\mathcal{F}_n\to\mathcal{F}$ in $L^1(0,T;\mathscr{H})$ and $U_n\to U$ in $C([0,T],\mathscr{H})$. ■

Definition 4.3. The energy $\mathscr{E}(t)$ of a solution $U(t)=(u(t),w(t),u_t(t),w_t(t))$ of (4.11) on $[0,T]$ is defined by

$$\mathcal{E}(t)=\frac{1}{2}\|U(t)\|_{\mathscr{H}}^2+\int_0^1 F(u,w)dx, \quad \forall\, t\in[0,T]. \tag{4.40}$$

■

Lemma 4.3. Under assumption $(\mathbf{A}_2)$, the energy $\mathcal{E}(t)$ of a solution $U(t) = (u(t), w(t), u_t(t), w_t(t))$ on $[0, T]$ satisfies the inequality

$$\frac{1}{2}\|U(t)\|^2_{\mathscr{H}} \leqslant \mathcal{E}(t) \leqslant \beta_0\big(1 + \|U(t)\|^{p+1}_{\mathscr{H}}\big), \quad \forall t \in [0, T]. \tag{4.41}$$

where β_0 is a constant independent of $U(t)$.

Proof. Since $F(u, w) \geqslant 0$, we immediately obtain

$$\frac{1}{2}\|U(t)\|^2_{\mathcal{H}} \leqslant \mathcal{E}(t). \tag{4.42}$$

Moreover, since $p \geqslant 1$, we have

$$\|U(t)\|^2_{\mathcal{H}} \leqslant (1 + \|U(t)\|^{p+1}_{\mathcal{H}}). \tag{4.43}$$

Now, using (3.6) and the embedding $H^1_0(0, 1) \hookrightarrow L^{p+1}(0, 1)$

$$\begin{aligned}\int_0^1 F(u, w)dx \leqslant C_F(1 + \|u\|^{p+1}_{p+1} + \|w\|^{p+1}_{p+1}) \leqslant & C_F(1 + \|u_x\|^{p+1} + \|w_x\|^{p+1}) \\ \leqslant & C_F(1 + \|U(t)\|^{p+1}_{*} + \|U(t)\|^{p+1}_{*}) \\ \leqslant & C_F\left(1 + \frac{2}{\gamma_0^{(p+1)/2}}\|U(t)\|^{p+1}_{\mathscr{H}}\right) \\ \leqslant & C_F \max\left\{1, \frac{2}{\gamma_0^{(p+1)/2}}\right\}\left(1 + \|U(t)\|^{p+1}_{\mathscr{H}}\right).\end{aligned} \tag{4.44}$$

From (4.42)–(4.44) we obtain (4.41), completing the proof of the lemma. □

Lemma 4.4. Under assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$, let $U(t) = (u(t), w(t), u_t(t), w_t(t))$ be a solution of (4.11) on $[0, T]$, then

$$\mathcal{E}(t) \leqslant \rho_0 e^{\frac{m}{2}t} \int_0^t \Big(\|h_1(s)\|^2 + \|h_2(s)\|^2\Big), \quad \forall\, t \in [0, T], \tag{4.45}$$

where

$$\rho_0 = \max\left\{\frac{1}{2(\rho_1 + 2)m}, \frac{(\rho_1 + 2)m}{2}\right\}. \tag{4.46}$$

Proof. Let $U(t)$ be a strong solution of (4.11). Multiplying (1.6) by u_t, (1.7) by w_t, integrating by parts each equation and summing the resulting equalities, we get

$$\frac{d\mathcal{E}}{dt}(t) = -\int_0^1 g_1(u_t)u_t dx - \int_0^1 g_2(w_t)w_t dx + \int_0^1 h_1(t)u_t dx + \int_0^1 h_2(t)w_t dx. \tag{4.47}$$

Using (3.2), we have

$$\begin{aligned} -\int_0^1 g_1(u_t)u_t dx \leqslant -m\|u_t\|^2, \\ -\int_0^1 g_2(w_t)w_t dx \leqslant -m\|w_t\|^2. \end{aligned} \tag{4.48}$$

Using Young's inequality

$$\begin{aligned}\int_0^1 h_1(t)u_t dx \leqslant \frac{1}{2(\rho_1 + 2)m}\|h_1\|^2 + \frac{(\rho_1 + 2)m}{2}\|u_t\|^2, \\ \int_0^1 h_2(t)w_t dx \leqslant \frac{1}{2(\rho_2 + 2)m}\|h_2\|^2 + \frac{(\rho_2 + 2)m}{2}\|w_t\|^2.\end{aligned} \tag{4.49}$$

From (4.47)–(4.49) we get

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt}(t) &\leqslant \frac{m}{2}\left(\rho_1\|u_t\|^2 + \rho_2\|w_t\|^2\right) + \frac{1}{2(\rho_1+2)m}\|h_1\|^2 + \frac{1}{2(\rho_2+2)m}\|h_2\|^2 \\
&\leqslant \frac{m}{2}\mathcal{E}(t) + \rho_0\big(\|h_1(t)\|^2 + \|h_2(t)\|^2\big),
\end{aligned} \tag{4.50}
$$

where ρ_0 is given by (4.46). Using Grönwall's Lemma and assumption $(\mathbf{A}_3)$, we obtain (4.45). Since F is continuous, it follows from Definition 4.2 that the inequality also holds for generalized solutions, thus concluding the proof. □

Theorem 4.1 (Global Existence and Uniqueness of Solution). Assume that assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold.

(a) If $U_0 \in D(\mathcal{A})$ and $h_i \in W^{1,1}_{loc}([0,\infty); L^2(0,1))$ for $i = 1,2$, then (4.11) admits a unique global strong solution $U : [0,\infty) \to \mathscr{H}$;

(b) If $U_0 \in \overline{D(\mathcal{A})} = \mathscr{H}$ and $h_i \in L^2_{loc}([0,\infty); L^2(0,1))$ for $i = 1,2$, then (4.11) admits a unique global generalized solution $U : [0,\infty) \to \mathscr{H}$.

Proof. Since $\mathcal{A}$ is linear, we have $\mathcal{A}0 = 0$, and since $\mathcal{B}$ is locally Lipschitz, **(a)** and **(b)** follow from [10, Theorem 7.2], and in both cases the solution is local, that is, $U : [0, T_{\max}) \to \mathscr{H}$. It follows from (4.45) that

$$
\lim_{t\uparrow T_{\max}} \|U(t)\|^2_{\mathscr{H}} \leqslant \rho_0 e^{\frac{m}{2}T_{\max}} \int_0^{T_{\max}} \left(\|h_1(s)\|^2 + \|h_2(s)\|^2\right) < \infty. \tag{4.51}
$$

Therefore, it follows again from [10, Theorem 7.2] that $T_{\max} = \infty$, and the proof of the theorem is complete. □

4.1 Absence of External Force

When we consider $h_i \equiv 0$, for $i = 1,2$, in (1.6)-(1.7) and (4.10)-(4.11), we can rewrite (1.6)-(1.9) as

$$
\begin{cases}
U_t(t) = -\mathcal{A}U(t) - \mathcal{B}(U(t)), \quad t > 0, \\
U(0) = U_0.
\end{cases} \tag{4.52}
$$

In this case, the existence and uniqueness of solutions can be obtained via the theory of semigroups of linear operators.

Definition 4.4 (Strong Solution of (4.52)**).** We say that $U : [0,T) \to \mathscr{H}$, with $0 < T \leqslant \infty$, is a strong solution of (4.52) if U is continuously differentiable, $U(0) = U_0$, $U(t) \in D(\mathcal{A})$ for all $t \in [0,T)$, and (4.52) is satisfied in $[0,T)$.

■

Definition 4.5 (Mild Solution of (4.52)**).** We say that a continuous function $U : [0,T) \to \mathscr{H}$, with $0 < T \leqslant \infty$, is a mild solution of (4.52) if $U(t)$ satisfies the following integral equation:

$$
U(t) = -T(t)U_0 - \int_0^t \mathcal{B}(U(s))\, ds, \quad \forall t \in [0,T), \tag{4.53}
$$

where $\{T(t)\}_{t\geqslant 0}$ is the C_0-semigroup of contractions generated by $\mathcal{A}$.

■

Remark 4.1. The energy of a (strong or mild) solution of (4.52) is also given by (4.40).

■

Lemma 4.5. Under assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_2)$, the energy $\mathscr{E}(t)$ of a solution $U(t) = (u(t), w(t), u_t(t), w_t(t))$ of (4.52) is non-increasing and satisfies (4.41).

Proof. We shall show only that $\mathcal{E}(t)$ is non-increasing. Assume $U(t)$ is a strong solution. Then, using a similar procedure to the one used to obtain (4.47) and (4.48), we get

$$\frac{d\mathcal{E}}{dt}(t) \leqslant -m(\|u_t\|^2 + \|w_t\|^2). \tag{4.54}$$

Thus, for any $0 \leqslant t_1 < t_2 \leqslant T$,

$$\mathcal{E}(t_2) - \mathcal{E}(t_1) \leqslant -m \cdot \int_{t_1}^{t_2} (\|u_t(s)\|^2 + \|w_t(s)\|^2) ds \leqslant 0. \tag{4.55}$$

The remainder of the proof is similar to the proof of Lemma 4.3. The lemma is proved. $\square$

Theorem 4.2 (Global Existence and Uniqueness of Solution of (4.52)**).** Assume that assumptions $(\mathbf{A}_1) - (\mathbf{A}_2)$ hold.

(a) If $U_0 \in D(\mathcal{A})$ then (4.52) has a unique (global) mild solution $U : [0, \infty) \to \mathscr{H}$;

(b) If $U_0 \in \overline{D(\mathcal{A})} = \mathscr{H}$ then the (global) mild solution obtained in item (a) is a strong solution.

(c) For any two solutions $U_1(t)$ and $U_2(t)$ of (4.52) there exists a constant α_0 depending on $U_1(0)$ and $U_2(0)$ such that

$$\|U_1(t) - U_2(t)\|_{\mathscr{H}} \leqslant e^{\alpha_0 t} \|U_1(0) - U_2(0)\|_{\mathscr{H}}, \quad \forall\, t > 0. \tag{4.56}$$

Proof. **(a)-(b)**

Since $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup and $\mathcal{B}$ is locally Lipschitz, the existence and uniqueness of a local solution $U : [0, T_{\max}) \to \mathscr{H}$ follow from [9, Propositions 4.3.3 and 4.3.9]. Moreover, from Lemma 4.5 we have

$$\lim_{t \uparrow T_{\max}} \|U(t)\|^2_{\mathscr{H}} \leqslant \mathcal{E}(0) < \infty. \tag{4.57}$$

Thus, it follows from [9, Theorem 4.3.4] that $T_{\max} = \infty$.

(c) Let $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ be the energies of $U_1(t)$ and $U_2(t)$, respectively. It follows from Lemma 4.5 that

$$\|U_1(t)\|_{\mathscr{H}} + \|U_2(t)\|_{\mathscr{H}} \leqslant \mathcal{E}_1(0) + \mathcal{E}_2(0) =: R. \tag{4.58}$$

Since $\mathcal{B}$ is locally Lipschitz and $\{T(t)\}_{t \geqslant 0}$ is a C_0-semigroup of contractions, it follows from (4.53) that

$$\|U_1(t) - U_2(t)\|_{\mathscr{H}} \leqslant \|U_1(0) - U_2(0)\|_{\mathscr{H}} + \alpha_0 \int_0^t \|U_1(s) - U_2(s)\|_{\mathscr{H}} ds, \tag{4.59}$$

where α_0 is a positive constant depending on R, and hence on $U_1(0)$ and $U_2(0)$. Applying Grönwall's Lemma to the inequality above yields (4.56), completing the proof of the theorem. $\square$

Remark 4.2. It follows from Theorem 4.2 the existence of a dynamical system $(\mathscr{H}, S(t))$ where $\{S(t)\}_{t \geqslant 0}$ is a C_0-semigroup of nonlinear operators on $\mathscr{H}$. In this case, $S(t) : \mathscr{H} \to \mathscr{H}$ is given by

$$S(t)U_0 = U(t) = (u(t), w(t), u_t(t), w_t(t)), \tag{4.60}$$

where $U(t)$ is a (mild) solution of (4.52).

5 Exponential Decay of the Energy

In this section, we will establish the exponential decay of the energy $\mathcal{E}(t)$ for the solutions of (1.6)-(1.9) in both the non-autonomous case ($h_i \not\equiv 0$) and the autonomous case ($h_i \equiv 0$).

5.0.1 Non-Autonomous System

Lemma 5.1. Under the hypotheses of Theorem 4.1, the energy $\mathcal{E}(t)$ of a strong solution $U(t) = (u(t), w(t), u_t(t), w_t(t))$ of (4.11) satisfies the following inequality:

$$\frac{d\mathcal{E}}{dt}(t) \leqslant -\frac{m}{2}\big(\|u_t\|^2 + \|w_t\|^2\big) + \frac{1}{2m}\big(\|h_1(t)\|^2 + \|h_2(t)\|^2\big). \tag{5.1}$$

Proof. Let $U(t)$ be a strong solution of (4.11). Multiplying (1.6) by u_t, (1.7) by w_t, integrating by parts each of the resulting equations and summing them, we obtain

$$\frac{d\mathcal{E}}{dt}(t) = -\int_0^1 g_1(u_t)u_t dx - \int_0^1 g_2(w_t)w_t dx + \int_0^1 h_1(t)u_t dx + \int_0^1 h_2(t)w_t dx. \tag{5.2}$$

Using (3.1) and Young's inequality, we arrive at (5.1), and the proof of the lemma is complete. □

Lemma 5.2. Under the hypotheses of Theorem 4.1, let $U(t) = (u(t), w(t), u_t(t), w_t(t))$ be a strong solution of (4.11) and let $\mathcal{I}(t)$ be the functional given by

$$\mathcal{I}(t) := \rho_1 \int_0^1 u_t u dx + \rho_2 \int_0^1 w_t w dx. \tag{5.3}$$

Then, $\mathcal{I}(t)$ satisfies

$$\frac{d\mathcal{I}}{dt}(t) \leqslant \beta_1\big(\|u_t\|^2 + \|w_t\|^2\big) - \frac{\sigma_0}{2}\big(\|u_x\|^2 + \|w_x\|^2\big) - \int_0^1 F(u,w)dx + \frac{1}{\sigma_0}\big(\|h_1\|^2 + \|h_2\|^2\big), \tag{5.4}$$

where

$$\beta_1 := \max\{\rho_1, \rho_2\} + \frac{M^2}{\sigma_0}. \tag{5.5}$$

Proof. Multiplying (1.6) by u, (1.7) by w, integrating by parts each of the equations and summing the results, we have

$$\begin{aligned} \frac{d}{dt}\big[\rho_1 u_t u + \rho_2 w_t w\big] &= \rho_1\|u_t\|^2 + \rho_2\|w_t\|^2 - \left[\left\|\sqrt{a_1}u_x + \frac{a_2}{\sqrt{a_1}}w_x\right\|^2 + \left(a_3 - \frac{a^2}{a_1}\right)\|w_x\|^2\right] \\ &- \int_0^1 \nabla F(u,w)\cdot(u,w)dx - \int_0^1 g_1(u_t)u dx - \int_0^1 g_2(w_t)w dx + \int_0^1 h_1 u dx + \int_0^1 h_2 w dx. \end{aligned} \tag{5.6}$$

Using the equivalence (4.8) and (3.4)

$$\begin{aligned} \frac{d}{dt}\big[\rho_1 u_t u + \rho_2 w_t w\big] &\leqslant \rho_1\|u_t\|^2 + \rho_2\|w_t\|^2 - \sigma_0\big(\|u_x\|^2 + \|w_x\|^2\big) \\ - \int_0^1 F(u,w)dx &- \int_0^1 g_1(u_t)u dx - \int_0^1 g_2(w_t)w dx + \int_0^1 h_1 u dx + \int_0^1 h_2 w dx. \end{aligned} \tag{5.7}$$

Using (3.2), Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} -\int_0^1 g_1(u_t)u dx &\leqslant \frac{M^2}{\sigma_0}\|u_t\|^2 + \frac{\sigma_0}{4}\|u_x\|^2, \\ -\int_0^1 g_2(w_t)w dx &\leqslant \frac{M^2}{\sigma_0}\|w_t\|^2 + \frac{\sigma_0}{4}\|w_x\|^2, \\ \int_0^1 h_1 u dx &\leqslant \frac{1}{\sigma_0}\|h_1\|^2 + \frac{\sigma_0}{4}\|u_x\|^2, \\ \int_0^1 h_2 w dx &\leqslant \frac{1}{\sigma_0}\|h_w\|^2 + \frac{\sigma_0}{4}\|w_x\|^2. \end{aligned} \tag{5.8}$$

From (5.6)–(5.8) we obtain (5.4) with β_1 given by (5.5), and the proof of the lemma is complete. □

Theorem 5.1. Under the assumptions of Theorem 4.1 and ($\mathbf{A}_4$), the energy $\mathcal{E}(t)$ of any solution $U(t) = (u(t), w(t), u_t(t), w_t(t))$ of (4.11) decays exponentially. More precisely, there exist positive constants ω, ω_1 and ω_2 such that for every solution $U(t)$, its energy satisfies

$$\mathcal{E}(t) \leqslant e^{-\omega t}\big(\omega_1 \mathcal{E}(0) + \omega_2 C_h\big), \quad \forall\, t \geqslant 0. \tag{5.9}$$

Proof. Let $U(t) = (u(t), w(t), u_t(t), w_t(t))$ be a strong solution of (4.11) and let $\mathcal{L}(t)$ be the functional given by

$$\mathcal{L}(t) = \mathcal{I}(t) + N\mathcal{E}(t), \tag{5.10}$$

where N is a positive constant to be fixed later. Note that, from the definitions of $\mathcal{I}$ and $\mathcal{E}$, there exists a positive constant c_0, independent of $U(t)$, such that

$$|\mathcal{L}(t) - N\mathcal{E}(t)| \leqslant c_0\mathcal{E}(t) \quad \Longleftrightarrow \quad (N - c_0)\mathcal{E}(t) \leqslant \mathcal{L}(t) \leqslant (N + c_0)\mathcal{E}(t). \tag{5.11}$$

Moreover, from (5.10), (5.1) and (5.4), we have

$$\begin{aligned} \frac{d\mathcal{L}}{dt}(t) \leqslant &-\left(\frac{Nm}{2} - \beta_1\right)\big(\|u_t\|^2 + \|w_t\|^2\big) - \sigma_0\big(\|u_x\|^2 + \|w_x\|^2\big) - \int_0^1 F(u, w)dx \\ &+ \left(\frac{1}{2m} + \frac{1}{\sigma_0}\right)\big(\|h_1\|^2 + \|h_2\|^2\big). \end{aligned} \tag{5.12}$$

Taking

$$N > \max\left\{\frac{2\beta_1}{m}, c_0\right\} \tag{5.13}$$

and using (4.8), we obtain positive constants α_1, α_2, c_1, α_4, independent of the solution $U(t)$, such that

$$\alpha_1\mathcal{E}(t) \leqslant \mathcal{L}(t) \leqslant \alpha_2\mathcal{E}(t) \quad \text{and} \quad \frac{d\mathcal{L}}{dt}(t) \leqslant -c_1\mathcal{E}(t) + \alpha_4\big(\|h_1\|^2 + \|h_2\|^2\big). \tag{5.14}$$

From the two previous inequalities, we obtain

$$\frac{d\mathcal{L}}{dt}(t) \leqslant -\alpha_3\mathcal{L}(t) + \alpha_4\big(\|h_1\|^2 + \|h_2\|^2\big) \quad \text{where} \quad \alpha_3 := \min\left\{\frac{c_1}{\alpha_2}, \sigma\right\} \tag{5.15}$$

and σ is given in (3.7). From (5.14) and (5.15), we conclude

$$\mathcal{E}(t) \leqslant e^{-\alpha_3 t}\left(\frac{\alpha_2}{\alpha_1}\mathcal{E}(0) + \frac{\alpha_4}{\alpha_1}\int_0^t e^{\alpha_1 s}\big(\|h_1(s)\|^2 + \|h_2(s)\|^2\big)ds\right) \leqslant e^{-\alpha_3 t}\left(\frac{\alpha_2}{\alpha_1}\mathcal{E}(0) + \frac{\alpha_4}{\alpha_1}C_h\right). \tag{5.16}$$

Using a density argument, the previous inequality also holds for every generalized solution in a compact interval $[0, T]$, and this concludes the proof of the theorem.

□

5.0.2 Autonomous System

Theorem 5.2. Under the hypotheses of Theorem 4.2, the energy $\mathcal{E}(t)$ of every solution $U(t) = (u(t), w(t), u_t(t), w_t(t))$ of (4.52) decays exponentially. More precisely, there exist positive constants κ and κ_1 such that for every solution $U(t)$, its energy $\mathcal{E}(t)$ satisfies

$$\mathcal{E}(t) \leqslant \kappa_1 e^{-\kappa t}\mathcal{E}(0), \quad \forall\, t \geqslant 0. \tag{5.17}$$

Proof. Let $U(t)$ be a strong solution of (4.52). Following the steps in the proof of Lemma 5.1, we obtain

$$\frac{d\mathcal{E}}{dt}(t) \leqslant -m\big(\|u_t\|^2 + \|w_t\|^2\big). \tag{5.18}$$

Following the steps in Lemma 5.2, we obtain

$$\frac{d\mathcal{I}}{dt}(t) \leqslant \beta_2\big(\|u_t\|^2 + \|w_t\|^2\big) - \frac{\sigma_0}{2}\big(\|u_x\|^2 + \|w_x\|^2\big) - \int_0^1 F(u,w)dx, \tag{5.19}$$

where $\mathcal{I}$ is given in (5.3) and

$$\beta_2 := \max\{\rho_1, \rho_2\} + \frac{M^2}{2\sigma_0}. \tag{5.20}$$

Define $\mathcal{L}_1(t)$ as

$$\mathcal{L}_1(t) = \mathcal{I}(t) + N_1\mathcal{E}(t), \tag{5.21}$$

where N_1 is a positive constant to be determined. There exists a positive constant c_1 such that

$$|\mathcal{L}_1(t) - N_1\mathcal{E}(t)| \leqslant c_1\mathcal{E}(t). \tag{5.22}$$

Then, taking

$$N > \max\left\{c_1, \frac{\beta_2}{M}\right\}, \tag{5.23}$$

we obtain positive constants κ and κ_1 satisfying (5.17), and the proof is complete. □

6 Determining Functionals for the Autonomous System

We now consider the dynamical system $(\mathscr{H}, S(t))$ arising from Theorem 4.2.

Let $U_i(t) = S(t)W_i = (u^i(t), w^i(t), u^i_t(t), w^i_t(t))$, for $i = 1, 2$, be two strong solutions of (4.52). Consider

$$U = U_1 - U_2 \quad \text{with} \quad U(t) = (u(t), w(t), u_t(t), w_t(t)), \tag{6.1}$$

where

$$u = u^1 - u^2, \quad \text{and} \quad w = w^1 - w^2. \tag{6.2}$$

Then, $U(t)$ satisfies

$$\rho_1 u_{tt} - a_1 u_{xx} - a_2 w_{xx} = G_1(u_t) + F_1(u,w), \tag{6.3}$$
$$\rho_2 w_{tt} - a_2 u_{xx} - a_3 w_{xx} = G_2(w_t) + F_2(u,w), \tag{6.4}$$

where

$$G_1(u_t) = -[g_1(u^1_t) - g_1(u^2_t)], \quad G_2(w_t) = -[g_2(w^1_t) - g_2(w^2_t)], \tag{6.5}$$

and

$$F_i(u,w) = -[f_i(u^1, w^1) - f_i(u^2, w^2)], \quad \text{for} \quad i = 1, 2. \tag{6.6}$$

Moreover,

$$u(0,t) = u(1,t) = w(0,t) = w(1,t) = 0, \quad t > 0. \tag{6.7}$$

The energy $E(t)$ of $U(t)$ is defined as

$$E(t) = \frac{1}{2}\|U(t)\|^2_{\mathscr{H}}. \tag{6.8}$$

Lemma 6.1. Under the hypotheses of Theorem 4.2, the energy $E(t)$ of $U(t)$ satisfies the following inequality:

$$\frac{dE}{dt}(t) \leqslant -\frac{m}{2}\big(\|u_t(t)\|^2 + \|w_t(t)\|^2\big) + K_0\big(\|u\|_{2q}^2 + \|w\|_{2q}^2\big), \tag{6.9}$$

where K_0 is a positive constant depending on $U_1(0)$ and $U_2(0)$.

Proof. Multiplying (1.6) by u_t, (1.7) by w_t, integrating by parts on $[0,1]$, and summing the results, we obtain

$$\frac{dE}{dt}(t) = \int_0^1 G_1(u_t)u_t dx + \int_0^1 G_2(w_t)w_t dx + \int_0^1 F_1(u,w)u_t dx + \int_0^1 F_2(u,w)w_t dx. \tag{6.10}$$

From (3.2) we have

$$\frac{dE}{dt}(t) \leqslant -m\big(\|u_t\|^2 + \|w_t\|^2\big) + \int_0^1 F_1(u,w)u_t dx + \int_0^1 F_2(u,w)w_t dx. \tag{6.11}$$

By Young's inequality, using reasoning similar to that used to obtain (4.34), and noting that $U_i(t)$ is bounded (see Lemma 4.5), we obtain

$$\begin{aligned}\int_0^1 F_1(u,w)u_t dx &\leqslant \frac{1}{2m}\int_0^1 |F_1(u,w)|^2 dx + \frac{m}{2}\|u_t\|^2\\ &\leqslant K_0\big(\|u\|_{2p}^2 + \|w\|_{2p}^2\big) + \frac{m}{2}\|u_t\|^2,\end{aligned} \tag{6.12}$$

where K_0 is a positive constant (generically represented) that depends on $U_1(0)$ and $U_2(0)$. Similarly, we also have

$$\int_0^1 F_2(u,w)w_t dx \leqslant K_0\big(\|u\|_{2p}^2 + \|w\|_{2p}^2\big) + \frac{m}{2}\|w_t\|^2. \tag{6.13}$$

From (6.10)–(6.13), we obtain (6.9), completing the proof. $\square$

Lemma 6.2. Under the hypotheses of Theorem 4.2, let $U_i(t) = S(t)W_i = (u^i(t), w^i(t), u_t^i(t), w_t^i(t))$, $i = 1,2$, be two mild solutions of (4.52). Then the following inequality holds:

$$\|S(t)W_1 - S(t)W_2\|_{\mathscr{H}}^2 \leqslant \theta_1 e^{-\theta_2 t}\|W_1 - W_2\|_{\mathscr{H}}^2 + K_0 \sup_{s\in[0,t]} [\mu_{\mathscr{V}}(u(s), w(s))]^2, \quad t \geqslant 0, \tag{6.14}$$

where

$$u = u^1 - u^2, \quad w = w^1 - w^2, \tag{6.15}$$

and $\mu_{\mathscr{V}}(\cdot)$ is the seminorm in $\mathscr{V}_1$ given by

$$\mu_{\mathscr{V}}(u,w) = (\|u\|_{2q}^2 + \|w\|_{2q}^2)^{1/2}, \quad \forall (u,w) \in \mathscr{V}_1, \tag{6.16}$$

θ_1, θ_2 are positive constants and $K_0 > 0$ is a constant depending on W_i, $i = 1,2$.

Proof. Let $S(t)U_i$ for $i = 1,2$ be strong solutions. Multiplying (1.6) by u, (1.7) by w, integrating by parts over $[0,1]$, summing the results, and using the equivalence (4.8), we obtain

$$\begin{aligned}&\frac{d}{dt}\big[\rho_1 u_t u + \rho_2 w_t w\big] \leqslant \rho_1\|u_t\|^2 + \rho_2\|w_t\|^2 - \sigma_0\big(\|u_x\|^2 + \|w_x\|^2\big)\\ &+ \int_0^1 G_1(u_t)u dx + \int_0^1 G_2(w_t)w dx + \int_0^1 F_1(u,w)u dx + \int_0^1 F_2(u,w)u dx.\end{aligned} \tag{6.17}$$

Using (3.2) and Young's and Poincaré's inequalities,

$$\begin{aligned}\int_0^1 G_1(u_t)udx &\leqslant \frac{M^2}{\sigma_0}\|u_t\|^2 + \frac{\sigma_0}{4}\|u_x\|^2,\\ \int_0^1 G_2(\varphi_t)\varphi dx &\leqslant \frac{M^2}{\sigma_0}\|\varphi_t\|^2 + \frac{\sigma_0}{4}\|\varphi_x\|^2.\end{aligned} \tag{6.18}$$

Now, applying Young's and Poincaré's inequalities, and using (3.5), we obtain

$$\begin{aligned}\int_0^1 F_1(u,w)udx &\leqslant \frac{1}{\sigma_0}\int_0^1 |F_1(u,w)|^2dx + \frac{\sigma_0}{4}\|u_x\|^2\\ \leqslant C\Bigg(\sum_{i=1}^{2}\Big(1+\|u^i\|_{2q}^{2(q-1)} + \|w^i\|_{2q}^{2(q-1)}\Big)\Bigg)&\big(\|u\|_{2q}^2 + \|w\|_{2q}^2\big) + \frac{\sigma_0}{4}\|u_x\|^2,\end{aligned} \tag{6.19}$$

where C is a generic positive constant. Since the solutions $U_i(t)$, $i=1,2$, are bounded, we have

$$\int_0^1 F_1(u,w)u\,dx \leqslant K_0\big(\|u\|_{2q}^2 + \|w\|_{2q}^2\big) + \frac{\sigma_0}{4}\|u_x\|^2, \tag{6.20}$$

where K_0 is a generic positive constant depending on W_i, for $i=1,2$. Similarly, we have

$$\int_0^2 F_2(u,w)wdx \leqslant K_0\big(\|u\|_{2q}^2 + \|w\|_{2q}^2\big) + \frac{\sigma_0}{4}\|w_x\|^2. \tag{6.21}$$

From (6.17)–(6.18) and (6.20)–(6.21), we obtain

$$\frac{d}{dt}\left[\rho_1\int_0^1 u_tu\,dx + \rho_2\int_0^1 w_twdx\right] \leqslant \vartheta_0\big(\|u_t\|^2+\|w_t\|^2\big) - \frac{\sigma_0}{2}\|(u,w)\|_\star^2 + K_0\Big(\|u\|_{2q}^2 + \|w\|_{2q}^2\Big), \tag{6.22}$$

where

$$\vartheta_0 := \max\{\rho_1,\rho_2\} + \frac{M^2}{\sigma_0}. \tag{6.23}$$

Using (4.8)

$$\frac{d}{dt}\left[\rho_1\int_0^1 u_tu\,dx + \rho_2\int_0^1 w_twdx\right] \leqslant \vartheta_0\big(\|u_t\|^2+\|w_t\|^2\big) - \frac{\sigma_0}{2\sigma_1}\|(u,w)\|_{\mathscr{V}_1}^2 + K_0\Big(\|u\|_{2q}^2 + \|w\|_{2q}^2\Big). \tag{6.24}$$

Consider the functional $\mathcal{J}(t)$ given by

$$\mathcal{J}(t) = \rho_1\int_0^1 u_tu\,dx + \rho_2\int_0^1 w_twdx + NE(t), \tag{6.25}$$

where N is a positive constant to be fixed later. Using Young's inequality and the equivalence (4.29), it is possible to obtain a positive constant $\tilde{c}_0$, independent of $U_1(t)$ and $U_2(t)$, such that

$$|\mathcal{J}(t) - NE(t)| \leqslant \tilde{c}_0E(t). \tag{6.26}$$

Moreover, from (6.9) and (6.24), we have

$$\frac{d\mathcal{J}}{dt}(t) \leqslant -\Big(\frac{Nm}{2} - \vartheta_0\Big)\big(\|u_t\|^2+\|w_t\|^2\big) - \frac{\sigma_0}{2\sigma_1}\|(u,w)\|_{\mathscr{V}_1}^2 + NK_0\Big(\|u\|_{2q}^2 + \|w\|_{2q}^2\Big). \tag{6.27}$$

Thus, choosing N large enough such that

$$N > \max\left\{N - c_0, \frac{2\vartheta_0}{m}\right\}, \tag{6.28}$$

we obtain positive constants η_0, η_1, and η_2 such that

$$\eta_0 E(t) \leqslant \mathcal{J}(t) \leqslant \eta_1 E(t) \tag{6.29}$$

and

$$\frac{d\mathcal{J}}{dt}(t) \leqslant -\eta_2 E(t) + K_0\big(\|u\|_{2q}^2 + \|w\|_{2q}^2\big). \tag{6.30}$$

From the two previous inequalities, it follows that

$$\begin{aligned} E(t) &\leqslant \frac{\eta_1}{\eta_0} E(0) e^{-\frac{\eta_2}{\eta_1}t} + K_0 e^{-\frac{\eta_2}{\eta_1}t} \int_0^t e^{\frac{\eta_2}{\eta_1}s} \big(\|u(s)\|_{2q}^2 + \|w(s)\|_{2q}^2\big) ds \\ &\leqslant \frac{\eta_1}{\eta_0} E(0) e^{-\frac{\eta_2}{\eta_1}t} + K_0 \sup_{s\in[0,t]} \big(\|u(s)\|_{2q}^2 + \|w(s)\|_{2q}^2\big). \end{aligned} \tag{6.31}$$

Note that

$$\mu_{\mathscr{V}}(u,w) = (\|u\|_{2q}^2 + \|w\|_{2q}^2)^{1/2} \tag{6.32}$$

is a seminorm in $\mathscr{V}_1$. Thus, we obtain (6.14), and by a density argument, the inequality (6.14) also holds for mild solutions, completing the proof of the lemma. □

Definition 6.1 (**Determining Set of Functionals**). Let $\mathscr{L} = \{l_j : j \in J\}$ be a set of continuous linear functionals on $\mathscr{V}$, where J is an index set. We say that $\mathscr{L}$ is a *determining set of functionals* for the trajectories of $(\mathscr{H}, S(t))$ if the following condition is satisfied:

For any two trajectories

$$S(t)W_i = \big(u^i(t), w^i(t), u_t^i(t), w_t^i(t)\big), \quad i = 1, 2, \tag{6.33}$$

the condition

$$\lim_{t\to\infty} \big|l_j\big(u^1(t), w^1(t)\big) - l_j\big(u^2(t), w^2(t)\big)\big|^2 = 0, \quad \forall j \in I, \tag{6.34}$$

implies that

$$\lim_{t\to\infty} \|S(t)W_1 - S(t)W_2\|_{\mathscr{H}}^2 = 0. \tag{6.35}$$

Definition 6.2. The completeness defect of a set $\mathscr{L} = \{l_j : j \in J\}$ of linear functionals on $\mathscr{V}_1$ with respect to a seminorm μ on $\mathscr{V}_1$ is the value

$$\epsilon_{\mathscr{L}}(\mathscr{V}_1, \mu) := \sup\{\mu(\mathbf{w}) : \mathbf{w} \in \mathscr{V}_1,\ l_j(\mathbf{w}) = 0,\ \forall j \in J,\ \|\mathbf{w}\|_{\mathscr{V}_1} \leqslant 1\}. \tag{6.36}$$

■

Lemma 6.3. Let $\epsilon_{\mathscr{L}} = \epsilon_{\mathscr{L}}(\mathscr{V}_1, \mu)$ be the completeness defect of a finite set $\mathscr{L} = \{l_j : j = 1, \ldots, N\}$ of linear functionals on $\mathscr{V}_1$ with respect to a seminorm μ on $\mathscr{V}_1$. Then, there exists a positive constant $C_{\mathscr{L}}$ such that

$$\mu(\mathbf{w}) \leqslant \epsilon_{\mathscr{L}} \|\mathbf{w}\|_{\mathscr{V}_1} + C_{\mathscr{L}} \cdot \max\{|l_j(\mathbf{w})| : j = 1, \ldots, N\}, \quad \forall\, \mathbf{w} \in \mathscr{V}_1. \tag{6.37}$$

Proof. Without loss of generality, we may assume that the functionals $\{l_j\}$ are linearly independent. This allows us to construct a biorthogonal system $\{e_j : j = 1, \ldots, N\} \subset \mathscr{V}_1$ for $\mathscr{L}$ (i.e., $l_j(e_i) = 0$ if $j \neq i$ and $l_j(e_j) = 1$). In this case, for any $\mathbf{w} \in \mathscr{V}_1$, the element $\mathbf{v} = \mathbf{w} - \sum_{i=1}^N l_i(\mathbf{w}) e_i$ satisfies $l_j(\mathbf{v}) = 0$ for $j = 1, \ldots, N$. By the definition in (6.36), we have

$$\mu(\mathbf{v}) \leq \epsilon_{\mathscr{L}} \|\mathbf{v}\|_{\mathscr{V}_1}. \tag{6.38}$$

It follows from the definition of $\mathbf{v}$ and the inequality above that

$$
\begin{aligned}
\mu(\mathbf{w}) &\leqslant \mu(\mathbf{w}-\mathbf{v}) + \mu(\mathbf{v}) \\
&\leqslant \mu\Big(\sum_{i=1}^{N} l_i(\mathbf{w}) e_i\Big) + \epsilon_{\mathscr{L}} \|\mathbf{v}\|_{\mathscr{V}_1} \\
&\leqslant \Big(\sum_{i=1}^{N} \mu(e_i)\Big) N \cdot \max_{1\leqslant i\leqslant N} \{|l_i(\mathbf{w})|\} + \epsilon_{\mathscr{L}} \Big\| \mathbf{w} - \sum_{i=1}^{N} l_i(\mathbf{w}) e_i \Big\|_{\mathscr{V}_1} \\
&\leqslant \Big(\sum_{i=1}^{N} \mu(e_i)\Big) N \cdot \max_{1\leqslant i\leqslant N} \{|l_i(\mathbf{w})|\} + \epsilon_{\mathscr{L}} \|\mathbf{w}\|_{\mathscr{V}_1} + \epsilon_{\mathscr{L}} \Big(\sum_{i=1}^{N} \|e_i\|_{\mathscr{V}}\Big) N \cdot \max_{1\leqslant i\leqslant N} \{|l_i(\mathbf{w})|\} \\
&= \epsilon_{\mathscr{L}} \|\mathbf{w}\|_{\mathscr{V}_1} + \Big[\sum_{i=1}^{N} \Big(\mu(e_i) + \epsilon_{\mathscr{L}} \|e_i\|_V\Big)\Big] N \cdot \max_{1\leqslant i\leqslant N} \{|l_i(\mathbf{w})|\}.
\end{aligned}
\tag{6.39}
$$

Setting

$$
C_{\mathscr{L}} := \Big[\sum_{i=1}^{N} \Big(\mu(e_i) + \epsilon_{\mathscr{L}} \|e_i\|_V\Big)\Big] N > 0, \tag{6.40}
$$

we obtain the result. □

Theorem 6.1 (Determining Functionals)**.** Under the hypotheses of Theorem (4.2), there exists a number $\varepsilon_0 > 0$ such that, if $\mathscr{L} = \{l_j : j = 1, \ldots, N\}$ is a set of linear and continuous functionals on $\mathscr{V}$ with $\varepsilon_{\mathscr{L},\mu_{\mathscr{V}}} \leqslant \varepsilon_0$, where $\varepsilon_{\mathscr{L},\mu_{\mathscr{V}}}$ is the completeness defect of $\mathscr{L}$ with respect to the norm $\mu_{\mathscr{V}}$ given in (6.16), then $\mathscr{L}$ is a determining set of functionals for $(\mathscr{H}, S(t))$.

Proof. Let $S(t)W_i = (u^i(t), w^i(t), u^i_t(t), w^i_t(t))$, for $i = 1, 2$, be two trajectories. We want to show that if $\varepsilon_{\mathscr{L},\mu_{\mathscr{V}}}$ is sufficiently small, then

$$
\lim_{t\to\infty} \big| l_j(u^1(t), w^1(t)) - l_j(u^2(t), w^2(t)) \big|^2 = 0, \quad j = 1, \ldots, N, \tag{6.41}
$$

implies that

$$
\lim_{t\to\infty} \|S(t)W_1 - S(t)W_2\|^2_{\mathscr{H}} = 0. \tag{6.42}
$$

Fixing $\tau > 0$, consider

$$
\mathscr{S}(t) := \sup_{s\in[t,t+\tau]} \max_{i\leqslant j\leqslant N} \Big| l_j(u^1(t), w^1(t)) - l_j(u^2(t), w^2(t)) \Big|^2. \tag{6.43}
$$

Condition (6.41) is equivalent to

$$
\lim_{t\to\infty} \mathscr{S}(t) = 0. \tag{6.44}
$$

Using (6.14), we have

$$
\begin{aligned}
\big\|S(t+\tau)W_1 - S(t+\tau)W_2\big\|^2_{\mathscr{H}} \leqslant{}& \theta_1 e^{-\theta_2\tau} \big\|S(t)W_1 - S(t)W_2\big\|^2_{\mathscr{H}} \\
&+ K_0 \sup_{s\in[t,t+\tau]} [\mu_{\mathscr{V}}(u(s), w(s))]^2, \quad t \geqslant 0,
\end{aligned}
\tag{6.45}
$$

where $u = u^1 - u^2$, $w = w^1 - w^2$ and K_0 is a positive constant depending on W_i, $i = 1, 2$.

Using Lemma 6.3 and Young's inequality, we have that for every $\delta > 0$ there exists a constant $C_{\mathscr{L},\delta} > 0$ such that

$$
\begin{aligned}
[\mu_{\mathscr{V}}(u(s), w(s))]^2 &\leqslant (1+\delta)\varepsilon^2_{\mathscr{L},\mu_{\mathscr{V}}} \|(u(s), w(s))\|^2_{\mathscr{V}_1} + C_{\mathscr{L},\delta} \cdot \max_{0\leqslant j\leqslant N} \{|l_j(u(s), w(s))|^2\} \\
&\leqslant (1+\delta)\varepsilon^2_{\mathscr{L},\mu_{\mathscr{V}}} \big\|S(s)W_1 - S(s)W_2\big\|^2_{\mathscr{H}} + C_{\mathscr{L},\delta} \cdot \max_{0\leqslant j\leqslant N} \{|l_j(u(s), w(s))|^2\}.
\end{aligned}
\tag{6.46}
$$

Taking the supremum over $s \in [t, t+\tau]$ on both sides of the inequality above, we get

$$\begin{aligned}\sup_{s\in[t,t+\tau]} [\mu_{\mathscr{V}}(u(s), w(s))]^2 \leqslant (1+\delta)\varepsilon^2_{\mathscr{L},\mu_{\mathscr{V}}} \sup_{s\in[t,t+\tau]} \big\|S(s)W_1 - S(s)W_2\big\|^2_{\mathscr{H}} \\ + C_{\mathscr{L},\delta} \cdot \sup_{s\in[t,t+\tau]} \max_{0\leqslant j\leqslant N} \{|l_j(u(s), w(s))|^2\}.\end{aligned} \tag{6.47}$$

From (4.56), we have

$$\sup_{s\in[t,t+\tau]} \big\|S(s)W_1 - S(s)W_2\big\|^2_{\mathscr{H}} \leqslant e^{2\alpha_0\tau}\big\|S(t)W_1 - S(t)W_2\big\|^2_{\mathscr{H}}, \tag{6.48}$$

where α_0 is a positive constant depending on W_i, $i = 1, 2$. From (6.47)-(6.48), we obtain

$$\sup_{s\in[t,t+\tau]} [\mu_{\mathscr{V}}(u(s), w(s))]^2 \leqslant (1+\delta)\varepsilon^2_{\mathscr{L},\mu_{\mathscr{V}}} e^{2\alpha_0\tau}\big\|S(t)W_1 - S(t)W_2\big\|_{\mathscr{H}} + C_{\mathscr{L},\delta}\mathscr{S}(t). \tag{6.49}$$

Combining (6.45) and (6.49), we get

$$\big\|S(t_0+\tau)W_1 - S(t_0+\tau)W_2\big\|^2_{\mathscr{H}} \leqslant \kappa_1\big\|S(t_0)W_1 - S(t_0)W_2\big\|^2_{\mathscr{H}} + C_{\mathscr{L}}\mathscr{S}(t_0), \tag{6.50}$$

where

$$\kappa_1 = \theta_1 e^{-\theta_2\tau} + K_0(1+\delta)\varepsilon^2_{\mathscr{L},\mu_{\mathscr{V}}} e^{2\alpha_0\tau} > 0. \tag{6.51}$$

Choosing δ and $\varepsilon_{\mathscr{L},\mu_{\mathscr{V}}}$ sufficiently small, we get $\kappa_1 < 1$. Thus, by induction, for every natural number n we have

$$\big\|S(t_0+n\tau)W_1 - S(t_0+n\tau)W_2\big\|^2_{\mathscr{H}} \leqslant \kappa_1^n\big\|S(t_0)W_1 - S(t_0)W_2\big\|^2_{\mathscr{H}} + C_{\mathscr{L}}\sum_{m=0}^{n-1}\kappa_1^{n-m-1}\mathscr{S}(t_0+m\tau). \tag{6.52}$$

Since limit (6.41) holds, so does (6.44), and the previous inequality implies

$$\lim_{n\to\infty} \|S(t_0+n\tau)W_1 - S(t_0+n\tau)W_2\|^2_{\mathscr{H}} = 0 \tag{6.53}$$

and the proof of the theorem is complete. □

7 Numerical approach

In this section, we developed an algorithm to obtain the numerical solution of a Nonlinear Non-Autonomous Binary Mixture Model. Here, we consider the following hypotheses: $g_i(s) = \gamma_i s$, where $\gamma_i \geq 0$ and $h_i \equiv 0$, with $i = 1, 2$. We will perform a series of numerical simulations to verify the properties of the theoretical results. In our numerical schemes, we consider an approximation by the finite element method in the spatial variable and the finite difference method in the temporal variable. For more details, see [21].

7.1 Variational formulation

Denoted by $(\cdot,\cdot)$ the inner product in $L^2(0,1)$ that is,

$$(f, g) = \int_0^1 fg\,dx, \quad \forall f, g \in L^2(0,1).$$

Thus, we can write our system (1.6)–(1.7) as follows:

Variational Problem: Find the functions $u(x,t)$ and $w(x,t)$ that satisfy the boundary and initial conditions (1.8)-(1.9), respectively, and

$$\rho_1(u_{tt}, v_1) + a_1(u_x, v_{1,x}) + a_2(w_x, v_{1,x}) + \gamma_1(u_t, v_1) + (f_1(u,w), v_1) = 0, \tag{7.1}$$

$$\rho_2(w_{tt}, v_2) + a_2(u_x, v_{2,x}) + a_3(w_x, v_{2,x}) + \gamma_2(w_t, v_2) + (f_2(u,w), v_2) = 0, \quad \forall t > 0. \tag{7.2}$$

with $v_1, v_2 \in H_0^1(0,1)$. For our purposes, it is necessary to introduce the following vector space: $\mathbb{V} := H_0^1(0,1) \times H_0^1(0,1)$. So, using a representation of the functions u, w in vector form, $\mathbf{u} = [u, w]^\top$, we introduce the following variational vector formulation.
Problem P: Find $\mathbf{u} : t \to \mathbb{V}$ such that

$$(\mathbf{u}_{tt}(t), \mathbf{v}) + \mathscr{A}(\mathbf{u}(t), \mathbf{v}) + \mathscr{F}(\mathbf{u}(t), \mathbf{v}) + \mathscr{C}(\mathbf{u}_t(t), \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbb{V}, \tag{7.3}$$

with $\mathbf{u}(t)$ satisfying the initial conditions

$$(\mathbf{u}(0), \mathbf{v}) = (\mathbf{u}_0, \mathbf{v}), \quad (\mathbf{u}_t(0), \mathbf{v}) = (\mathbf{u}_1, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{V}. \tag{7.4}$$

The functionals $\mathscr{A}, \mathscr{C}$, and $\mathscr{F} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ are given by

$$\begin{aligned}
\mathscr{A}(\mathbf{u}(t), \mathbf{v}) &= a_1(u_x, v_{1,x}) + a_2(w_x, v_{1,x}) + a_2(u_x, v_{2,x}) + a_3(w_x, v_{2,x}), \\
\mathscr{F}(\mathbf{u}(t), \mathbf{v}) &= (f_1(u, w), v_1) + (f_2(u, w), v_2), \\
\mathscr{C}(\mathbf{u}_t(t), \mathbf{v}) &= \gamma_1(u_t, v_1) + \gamma_2(w_t, v_2),
\end{aligned}$$

and

$$(\mathbf{u}_{tt}(t), \mathbf{v}) = \rho_1(u_{tt}, v_1) + \rho_2(w_{tt}, v_2),$$

where v_1 and v_2 are the components of the vector $\mathbf{v}$.

7.2 Algorithms

To obtain the full discretization of the problem (7.3). Firstly, we consider a partition X_h over the interval $\Omega = (0, 1)$, that is,

$$X_h = \{0 = x_0 < x_1 < \cdots < x_N = 1\}, \quad \Omega_{e+1} = (x_e, x_{e+1}),$$

where $\Omega_e \cap \Omega_{e'} = \varnothing$ for $e \neq e'$, and

$$\Omega = \bigcup_{e=1}^{N_e} \overline{\Omega}_e,$$

where N_e is the number of elements in the partition. We consider the following finite-dimensional subspaces:

$$S_1^h = \left\{ v \in C(0,1)\,;\; v\big|_{\Omega_e} \in P_1(\Omega_e) \right\},$$

where P_1 is the set of linear polynomials defined on Ω_e, and

$$\mathbb{V}^h = S_1^h \cap [H_0^1(0,1) \times H_0^1(0,1)].$$

We use a representation of the functions u^h, w^h in vector form $[u^h, w^h]^\top$. Thus, we can represent a typical element $\mathbf{u}^h \in \mathbb{V}^h$ by

$$\mathbf{u}^h(t, x) = \sum_{i=1}^{2N_e} d_i(t) \Phi_i(x),$$

where $2N_e$ is the total number of degrees of freedom of the finite element approximation, and $\Phi_i(x)$, $i = 1, \ldots, 2N_e$, are the global vector interpolation functions. Therefore, the semi-discrete finite element approximation of the variational problem (7.1) is characterized by the following:
Problem P^h: Find $\mathbf{u}^h : t \to \mathbb{V}^h$ such that

$$(\mathbf{u}_{tt}^h(t), \mathbf{v}^h) + \mathscr{A}(\mathbf{u}^h(t), \mathbf{v}^h) + \mathscr{F}(\mathbf{u}^h(t), \mathbf{v}^h) + \mathscr{C}(\mathbf{u}_t^h(t), \mathbf{v}^h) = 0, \quad \forall \mathbf{v}^h \in \mathbb{V}^h, \tag{7.5}$$

with $\mathbf{u}^h(t)$ satisfying the initial conditions

$$(\mathbf{u}^h(0), \mathbf{v}^h) = (\mathbf{u}_0^h, \mathbf{v}^h), \quad (\mathbf{u}_t^h(0), \mathbf{v}^h) = (\mathbf{u}_1^h, \mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbb{V}^h. \tag{7.6}$$

7.2.1 The fully discrete method

From Problem $\mathbf{P}^h$, we obtain the following nonlinear dynamical system in $\mathbb{R}^{2N_e}$:
Problem $\mathbf{P}_d^h$: Find $\mathbf{d} : [0,T] \to \mathbb{R}^{2N_e}$ such that

$$\mathbf{M}\ddot{\mathbf{d}}(t) + \mathbf{C}\dot{\mathbf{d}}(t) + \mathbf{K}\mathbf{d}(t) = \mathbf{F}(\mathbf{d}(t)), \tag{7.7}$$

with $\mathbf{d}(t)$ satisfying the initial conditions

$$\mathbf{d}(0) = \mathbf{d}_0, \quad \dot{\mathbf{d}}(0) = \mathbf{d}_1, \tag{7.8}$$

where $\mathbf{M}$ is the consistent mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the consistent elastic stiffness matrix, $\mathbf{F}(\mathbf{d}(t))$ is the vector of consistent generalized nodal forces at time t, and $\mathbf{d}(t)$ is the vector of generalized nodal displacements at time t.

To solve this system, we make a partition P over the interval $[0,T]$ of length Δt, such that $0 = t_0 < t_1 < \cdots < t_M = T$, with $t_{n+1} - t_n = \Delta t$, and we use the well-known Newmark method [30]. In our work, the Newmark scheme becomes

$$\begin{aligned}
\mathbf{M}\ddot{\mathbf{d}}_{n+1} + \mathbf{C}\dot{\mathbf{d}}_{n+1} + \mathbf{K}\mathbf{d}_{n+1} &= \mathbf{F}(\mathbf{d}_{n+1}), &(7.9)\\
\mathbf{d}_{n+1} &= \mathbf{d}_n + \Delta t\dot{\mathbf{d}}_n + \frac{\Delta t^2}{2}\big[(1-2\beta)\ddot{\mathbf{d}}_n + 2\beta\ddot{\mathbf{d}}_{n+1}\big], &(7.10)\\
\dot{\mathbf{d}}_{n+1} &= \dot{\mathbf{d}}_n + \Delta t\big[(1-\gamma)\ddot{\mathbf{d}}_n + \gamma\ddot{\mathbf{d}}_{n+1}\big], &(7.11)
\end{aligned}$$

where β and γ are two parameters that govern the stability and accuracy of the method.

To compute the numerical solution of the nonlinear discrete system (7.9)–(7.11), we use the following iterative procedure at each time step. In the Newmark method, we choose the displacement $\mathbf{d}_n$ as the unknown at time t_n. Then, we have

$$\begin{aligned}
\Big[\frac{\mathbf{M}}{\beta\Delta t^2} + \gamma\frac{\mathbf{C}}{\beta\Delta t} + \mathbf{K}\Big]\mathbf{d}_{n+1}^{\ell} &= \mathbf{M}\Big[\frac{1}{\beta\Delta t^2}\mathbf{d}_n + \frac{1}{\gamma\Delta t}\dot{\mathbf{d}}_n + \Big(1 - \frac{1}{2\beta}\Big)\ddot{\mathbf{d}}_n\Big]\\
&+ \mathbf{C}\Big[\frac{\gamma}{\beta\Delta t}\mathbf{d}_n + \Big(\frac{\gamma}{\beta} - 1\Big)\dot{\mathbf{d}}_n + \Delta t\Big(\frac{\gamma}{2\beta} - 1\Big)\ddot{\mathbf{d}}_n\Big] + \mathbf{F}(\mathbf{d}_{n+1}^{\ell-1}),\\
\ddot{\mathbf{d}}_{n+1} &= \frac{\mathbf{d}_{n+1} - \mathbf{d}_n}{\beta\Delta t^2} - \frac{\dot{\mathbf{d}}_n}{\beta\Delta t} - \Big(\frac{1}{2\beta} - 1\Big)\ddot{\mathbf{d}}_n,\\
\dot{\mathbf{d}}_{n+1} &= \frac{\gamma}{\beta\Delta t^2}(\mathbf{d}_{n+1} - \mathbf{d}_n) - \Big(\frac{\gamma}{\beta} - 1\Big)\dot{\mathbf{d}}_n - \Delta t\Big(\frac{\gamma}{2\beta} - 1\Big)\ddot{\mathbf{d}}_n,
\end{aligned} \tag{7.12}$$

with $\ell = 1, 2, 3, \ldots$

Since the matrices are symmetric and positive definite, the system above can be solved for any ℓ. Consequently, at each time step, we compute the solution $\mathbf{u}^{h,n+1}$ by solving a linear system. The iterative process given by (7.12) was stopped when the difference between successive iterates was less than or equal to 1.0×10^{-9}, and the total number of iterations ℓ was less than or equal to 20.

Remark 7.1. Since we are using interpolation polynomials of degree less than or equal to 1, the vector components $\mathbf{F}(\mathbf{d}_{n+1})$ are obtained by Gaussian Quadrature using three points.

7.3 Numerical simulation

In the sequel, we perform some numerical experiments based on the previous results. Firstly, we consider

$$f_1(u,w) = u^3 + uw^2 \quad \text{and} \quad f_2(u,w) = w^3 + u^2w.$$

Under these hypotheses, we have

$$\frac{dE}{dt} = -\gamma_1 \int_0^1 |u_t|^2\,dx - \gamma_2 \int_0^L |w_t|^2\,dx,$$

where

$$E(t) = \frac{1}{2}\int_0^1 \Bigg[\rho_1|u_t|^2 + \rho_2|w_t|^2 + \Big|\sqrt{a_1}u_x + \frac{a_2}{\sqrt{a_1}}w_x\Big|^2 + \Big(a_3 - \frac{a_2^2}{a_3}\Big)|w_x|^2 + \frac{1}{2}|u|^4 + \frac{1}{2}|w|^4 + |uw|^2\Bigg]dx.$$

Remark 7.2. To obtain the computational results, we used a code implemented in the C programming language. The graphics were generated using GNUplot. In all experiments, we considered the parameters of Newmark's algorithm $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$.

7.3.1 Experiment 1. (Conservative case: $\gamma_1 = 0$ and $\gamma_2 = 0$)

In this experiment, we consider a mixture problem with the following parameters: $\rho_1 = \rho_2 = 1.0$, $a_1 = 1.0$, $a_2 = 0.5$, and $a_3 = 2.0$. The initial conditions are:

$$u(x, 0) = -x + 2x^3 - x^4, \quad u_t(x, 0) = 0, \quad w(x, 0) = 2x - 6x^2 + 4x^3, \quad w_t(x, 0) = 0.$$

Furthermore, we used a finite element mesh with 800 elements and a time step $\Delta t = 10^{-4}$ s.

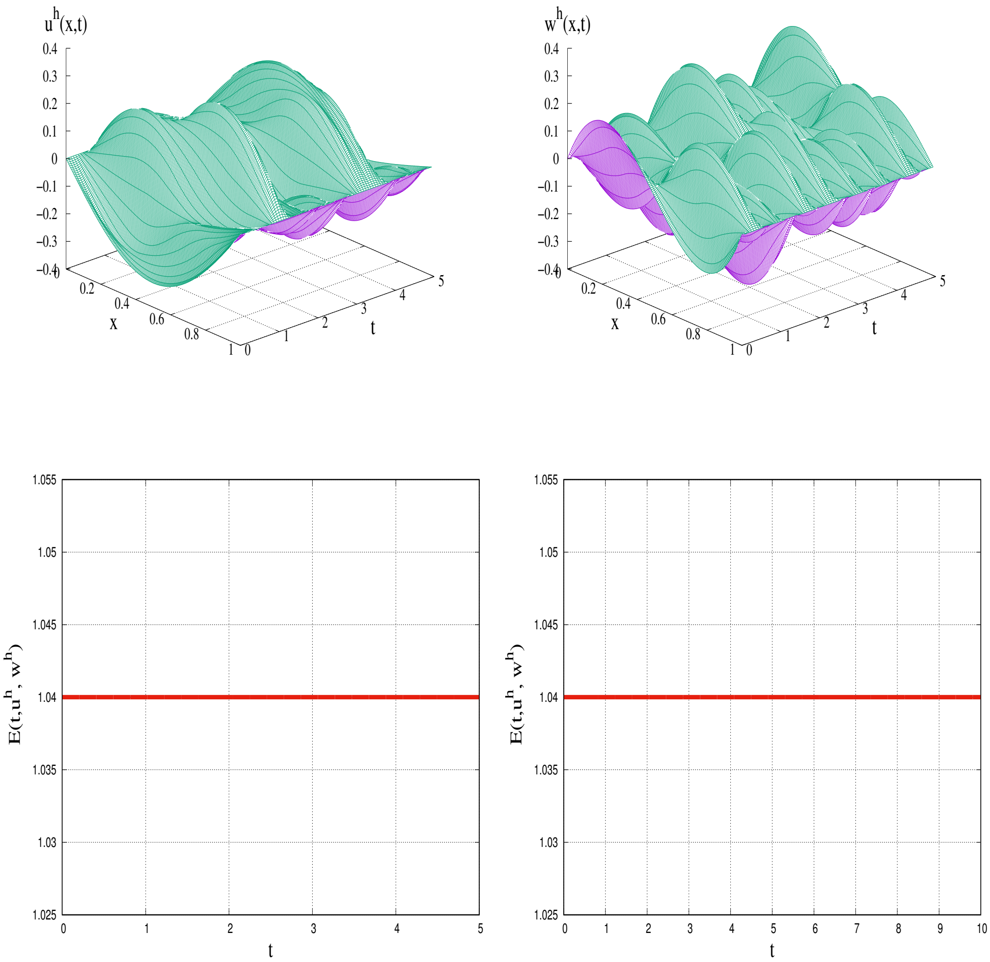

Figure 1: Evolution of the numerical solutions $u^h(x, t)$ and $w^h(x, t)$ at time 5 s, respectively. Numerical energy at 5 s and 10 s, respectively.

7.3.2 Experiment 2. (*Dissipative case with* $\gamma_1 = 0.5$ *and* $\gamma_2 = 0.6$)

In this experiment, we consider a mixture problem with the following parameters: $\rho_1 = \rho_2 = 1.0$, $a_1 = 1.0$, $a_2 = 0.5$, and $a_3 = 2.0$. The initial conditions are:

$$u(x, 0) = -x + 2x^3 - x^4, \quad u_t(x, 0) = 0, \quad w(x, 0) = 2x - 6x^2 + 4x^3, \quad w_t(x, 0) = 0.$$

Furthermore, we used a finite element mesh with 800 elements and a time step $\Delta t = 10^{-4}$ s.

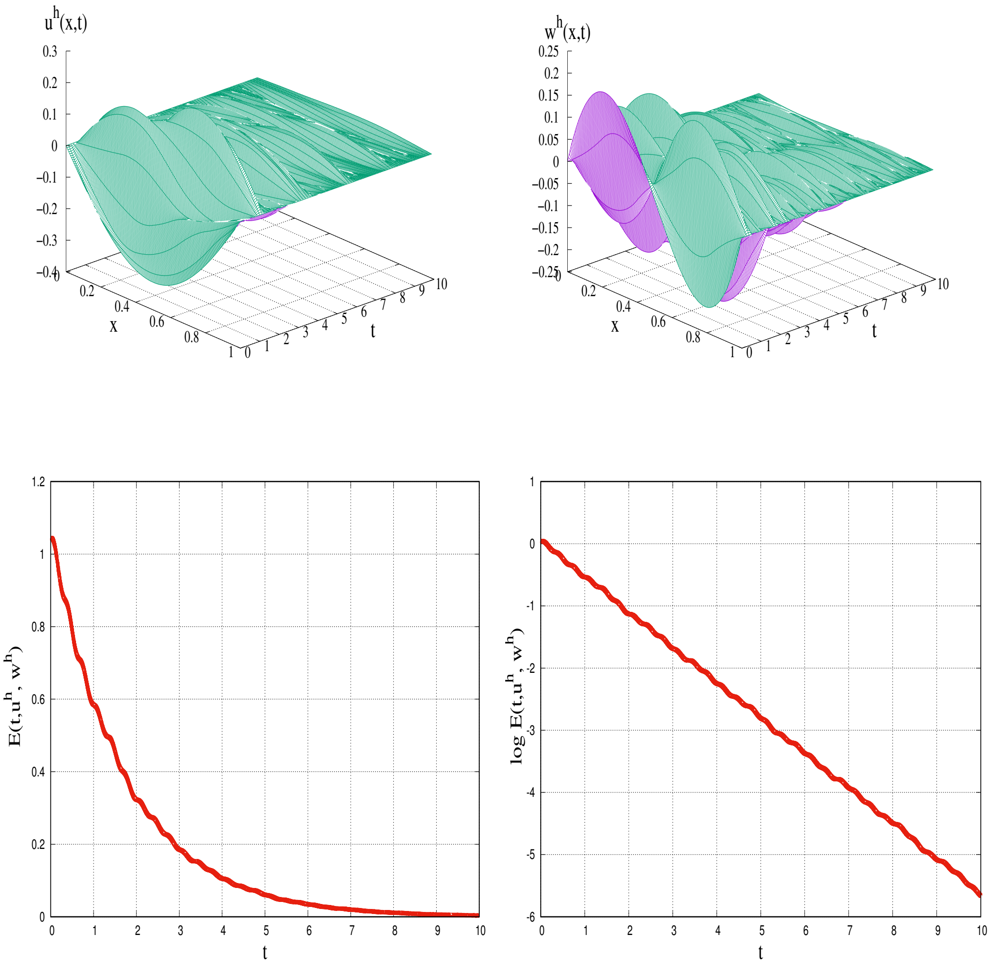

Figure 2: Evolution of the numerical solutions $u^h(x,t)$ and $w^h(x,t)$, respectively. Numerical energy behavior and its semilogarithmic scale at time 10 s, respectively. In this case, we obtained an exponential decay rate equal to -0.567079.

7.3.3 Experiment 3. (*Dissipative case with* $\gamma_1 = 1.0$ *and* $\gamma_2 = 1.0$)

Here, we consider a mixture problem with the followig parameters: $\rho_1 = \rho_2 = 1.0$, $a_1 = 1.0$, $a_2 = 0.5$ e $a_3 = 2.0$. The initial conditions:

$$u(x,0) = \sin 2\pi x, u_t(x,0) = \sin \pi x, \psi(x,0) = \sin \pi x \; w_t(x,0) = \sin 3\pi x.$$

Futhermore, a finite element mesh with 800 elements and $\Delta t = 10^{-4}$s.

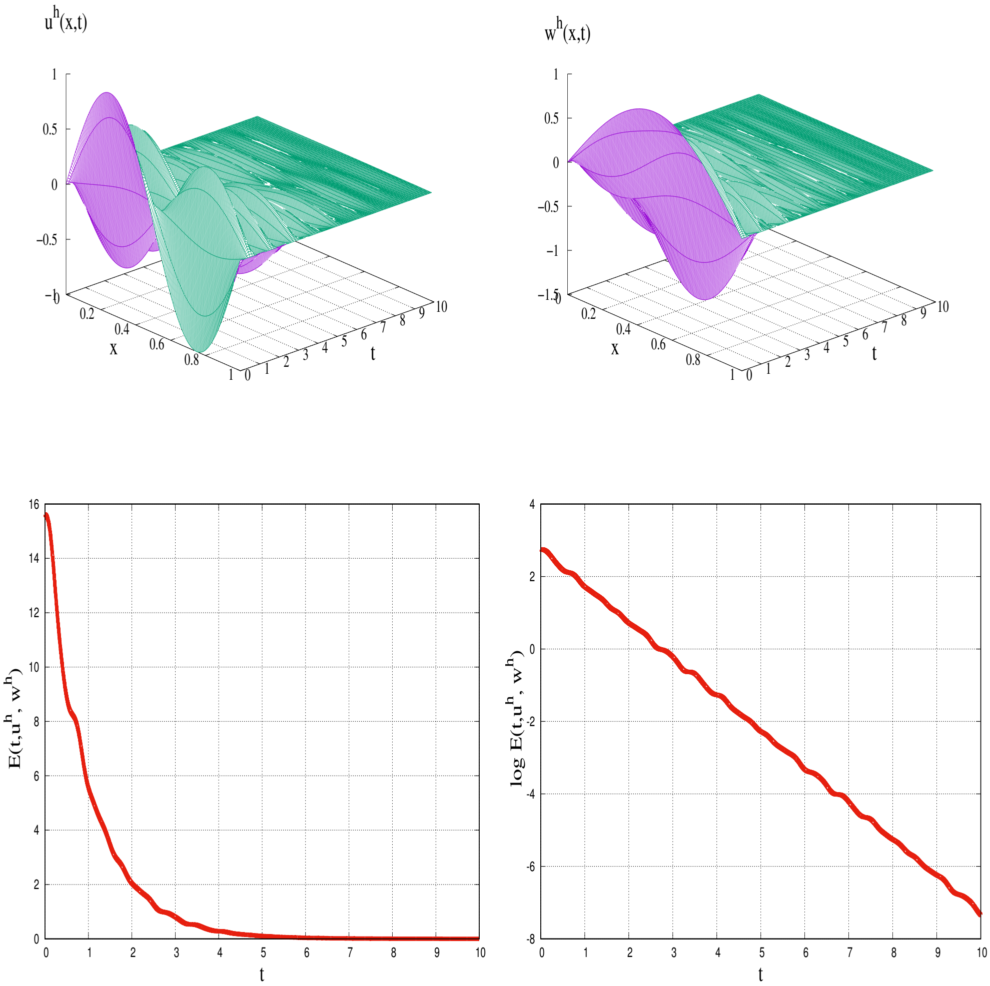

Figure 3: Evolution of the numerical solutions $u^h(x,t)$ and $w^h(x,t)$, respectively. Numerical energy behavior and its semilogarithmic scale at time 10 s, respectively. In this case, we obtained an exponential decay rate equal to -1.00091.

Remark 7.3. To obtain the exponential decay rate in the logarithmic versus time plots of Experiments 2 and 3, we used the `fit` command in the GNUplot environment.

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References

[1] M. S. Alves, J. E. Muñoz Rivera, and R. Quintanilla. Exponential decay in a thermoelastic mixture of solids. *International Journal of Solids and Structures*, 46(7):1659–1666, 2009.

[2] M. S. Alves, J. E. Muñoz Rivera, M. Sepúlveda, and O. P. V. Villagrán. Analyticity of semigroups associated with thermoviscoelastic mixtures of solids. *Journal of Thermal Stresses*, 32(10):986 – 1004, 2009.

[3] M. S. Alves, Jaime E. Muñoz Rivera, M. Sepúlveda, and O. P. Villagrán. Stabilization of mixture of two rigid solids modeling temperature and porosity. *Applied Mathematics Letters*, 25(5):884 – 889, 2012.

[4] R. J. Atkin and R. E. Craine. Continuum theories of mixtures: basic theory and historical development. *Quarterly Journal of Mechanics and Applied Mathematics*, 29(2):209–243, 1976.

[5] V. Barbu. *Nonlinear Differential Equations of Monotone Types in Banach Spaces.* Springer Monographs in Mathematics. Springer New York, 2010.

[6] A. Bedford and D. S. Drumheller. Theory of immiscible and structured mixtures. *International Journal of Engineering Science*, 21(8):863–960, 1983.

[7] R. M. Bowen. Part I - theory of mixtures. In A. CEMAL ERINGEN, editor, *Continuum Physics - Mixtures and EM field Theories*, volume 3, pages 1–127. Academic Press, 1976.

[8] R. M. Bowen and J. C. Wiese. Diffusion in mixtures of elastic materials. *Internat. J. Engrg. Sci.*, 7(7):689–722, 1969.

[9] T. Cazenave and A. Haraux. *An Introduction to Semilinear Evolution Equations.* Oxford lecture series in mathematics and its applications. Clarendon Press, 1998.

[10] I. Chueshov, M. Eller, and I. Lasiecka. On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation. *Communications in Partial Differential Equations*, 27(9-10):1901–1951, 2002.

[11] I. Chueshov and I. Lasiecka. *Von Karman Evolution Equations: Well-posedness and Long Time Dynamics.* Springer Monographs in Mathematics. Springer New York, 2010.

[12] F. Dell'Oro and J. E. Muñoz Rivera. Stabilization of ternary mixtures with frictional dissipation. *Asymptotic Analysis*, 89(3-4):235 – 262, 2014.

[13] A. C. Eringen and J. D. Ingram. A continuum theory of chemically reacting media–i. *International Journal of Engineering Science*, 3:197–212, 1965.

[14] C. Foias and G. Prodi. Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2. *Rendiconti del Seminario Matematico della Università di Padova*, 39:1–34, 1967.

[15] M. M. Freitas, T. A. Apalara, A. J. A. Ramos, and R. S. Costa. Attractors for partially damped systems of binary mixtures of solids. *Applicable Analysis*, 102(18):5103 – 5122, 2023.

[16] M. M. Freitas, A. L. C. Costa, and G. M. Araújo. Pullback dynamics of a non-autonomous mixture problem in one dimensional solids with nonlinear damping. *Communications on Pure and Applied Analysis*, 19(2):785 – 809, 2020.

[17] M. M. Freitas, M. J. Dos Santos, D. S. Almeida, M. L. Santos, and A. J. A. Ramos. Singular limits of binary mixtures in solids theory. *Journal of Mathematical Analysis and Applications*, 526(1), 2023.

[18] O. Gonzalez and A.M. Stuart. *A First Course in Continuum Mechanics*. A First Course in Continuum Mechanics. Cambridge University Press, 2008.

[19] A. E. Green and P. M. Naghdi. A dynamical theory of interacting continua. *International Journal of Engineering Science*, 3(2):231–241, 1965.

[20] A. E. Green and P. M. Naghdi. A note on mixtures. *International Journal of Engineering Science*, 6(11):631–635, 1968.

[21] T.J.R. Hughes. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. Dover Civil and Mechanical Engineering. Dover Publications, 2003.

[22] D. Ieşan. On the theory of mixtures of thermoelastic solids. *Journal of Thermal Stresses*, 14(4):389–408, 1991.

[23] J. D. Ingram and A. C. Eringen. A continuum theory of chemically reacting media—ii. constitutive equations of reacting fluid mixtures. *International Journal of Engineering Science*, 5:289–322, 1967.

[24] P. D. Kelly. A reacting continuum. *International Journal of Engineering Science*, 2(2):129–153, 1964.

[25] O. A. Ladyzhenskaya. A dynamical system generated by the navier-stokes equations. *Journal of Soviet Mathematics*, 3(4):458–479, 1975.

[26] F. Martínez and R. Quintanilla. Some qualitative results for the linear theory of binary mixtures of thermoelastic solids. *Collectanea Mathematica*, 46(3):263–277, 1995.

[27] L. G. R. Miranda, M. M. Freitas, and M. J. Dos Santos. Dynamics and singular limit for a swelling porous elastic soil model with fluid saturation and fractional delay. *Journal of Mathematical Analysis and Applications*, 545(2):129193, 2025.

[28] I. Müller. A thermodynamic theory of mixtures of fluids. *Archive for Rational Mechanics and Analysis*, 20:1–39, 1965.

[29] J. E. Muñoz Rivera, M. G. Naso, and R. Quintanilla. Decay of solutions for a mixture of thermoelastic solids with different temperatures. *Computers and Mathematics with Applications*, 71(4):991 – 1009, 2016.

[30] Nathan M. Newmark. A method of computation for structural dynamics. *Journal of the Engineering Mechanics Division*, 85(3):67–94, 1959.

[31] K. R. Rajagopal and A. S. Wineman. *Developments in the mechanics of interactions between a fluid and a highly elastic solid, volume II*, volume 217 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Essex, 1990.

[32] J.N. Reddy. *An Introduction to Continuum Mechanics*. Cambridge University Press, 2007.

[33] M. L. Santos, M. M. Freitas, and A. J. A. Ramos. Blow-up result and energy decay rates for binary mixtures of solids with nonlinear damping and source terms. *Nonlinear Analysis: Real World Applications*, 52:103026, 2020.

[34] R. E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Mathematical Surveys and Monographs. American Mathematical Society, 2013.

[35] C. A. Truesdell and R. A. Toupin. The classical field theories. In S. Flügge, editor, *Handbuch der Physik*, volume III/1, pages 226–793. Springer-Verlag, Berlin, 1960.